NATIONAL RESEARCH UNIVERSITY

Graph partitioning algorithms

Network Science
Lecture 7

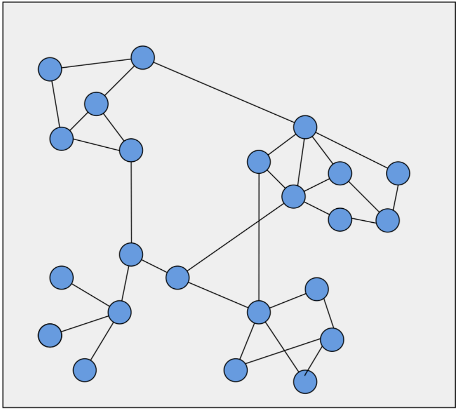
Leonid Zhukov

lzhukov@hse.ru
www.leonidzhukov.net/hse/2022/networks

National Research University Higher School of Economics
School of Data Analysis and Artificial Intelligence, Department of Computer Science

Graph partition

Graph partitioning is the split of a graph into subgraphs by partitioning its set of nodes into mutually exclusive groups



Graph partitioning

- Exact solution NP-hard
- Number of ways to divide network of *n* nodes in 2 groups (bi-partition): $2^{N-1} - 1$ ways - distinct cuts
- Combinatorial optimization problem:
  - optimization criterion
  - optimization method
- Solved by greedy, approximate algorithms or heuristics
- Balanced vs unbalanced partition
- 2 way-partiton vs multiway paritition

Graph partitioning algorithms

- Greedy optimization:
  Local search [Kernighan and Lin, 1970], [Fidducia and Mettheyses, 1982]
- Approximate optimization:
  Spectral graph partitioning [M. Fiedler, 1972], [Pothen et al 1990], [Shi and Malik, 2000]
  Multicommodity flow [Leighton and Rao, 1988]
- Randomized algorithms:
  Randomized min cut [D. Karger, 1993]
- Heuristics algorithms:
  Multilevel graph partitioning (METIS) [G. Karypis, Kumar 1998]

Optimization criterion

Graph $G(E, V)$ partition: $V = V_1 + V_2$

- Graph cut

$$Q = cut(V_1, V_2) = \sum_{i \in V_1, j \in V_2} e_{ij}$$

- Ratio cut:

$$Q = \frac{cut(V_1, V_2)}{||V_1||} + \frac{cut(V_1, V_2)}{||V_2||}$$

- Normalized cut:

$$Q = \frac{cut(V_1, V_2)}{Vol(V_1)} + \frac{cut(V_1, V_2)}{Vol(V_2)}$$

- Quotient cut (conductance):

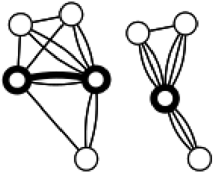
$$Q = \frac{cut(V_1, V_2)}{\min\left(Vol(V_1), Vol(V_2)\right)}$$

where: $Vol(V_1) = \sum_{i \in V_1, j \in V} e_{ij} = \sum_{i \in V_1} k_i$

Randomized min cut

Karger's algorithm for finding minimum cut

- Edge contraction - removing an edge and merging two vertices that it previously joined



- $P(\textit{final cut} = \textit{mincut}) \geq 2/n^2$
- Run it $\Omega(n^2)$ times and choose the smallest cut

Karger min cut

Input: Graph **G**

Output: class indicator vector *mincutX*

$X = \inf$

for $i = 1 : N$

  $X = GuessMinCut(G)$

  if $|X| < mincutk$

  $mincutX = X, mincutk = |X|$

return *mincutX*

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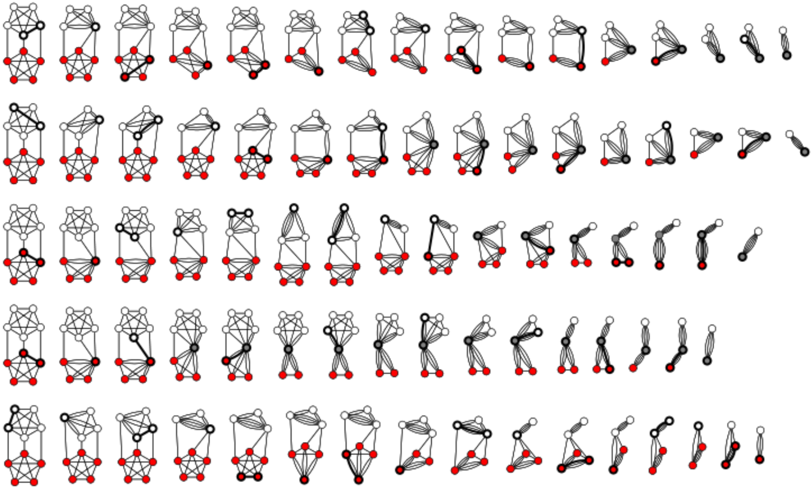
for $i = N$ down to 2

  pick a random edge e in G

  $G = G/e$

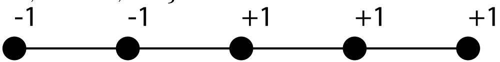
return the only cut in G

Karger's algoirthm



Linear algebra of graph cuts

- Let $V = V^+ + V^-$ be partitioning of the nodes
- Let $\mathbf{s} = \{+1, -1, +1, ... -1, +1\}^T$ - indicator vector



$$s(i) = \begin{cases} +1 : & \text{if } v(i) \in V^+ \\ -1 : & \text{if } v(i) \in V^- \end{cases}$$

- Number of edges, connecting V^+ and V^-

$$cut(V^+, V^-) = \frac{1}{4} \sum_{e(i,j)} (s(i) - s(j))^2 = \frac{1}{8} \sum_{i,j} A_{ij} (s(i) - s(j))^2 =$$

$$= \frac{1}{4} \sum_{i,j} (k_i \delta_{ij} s(i)^2 - A_{ij} s(i) s(j)) = \frac{1}{4} \sum_{i,j} (k_i \delta_{ij} - A_{ij}) s(i) s(j)$$

$$cut(V^+, V^-) = \frac{1}{4} \sum_{i,j} (D_{ij} - A_{ij}) s(i) s(j)$$

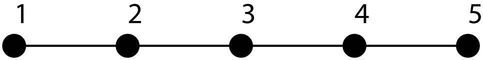
Graph cuts

- Graph Laplacian: $\mathbf{L}_{ij} = \mathbf{D}_{ij} - \mathbf{A}_{ij}$, where $\mathbf{D}_{ii} = diag(k_i)$

$$\mathbf{L}_{ij} = \begin{cases} k(i)\ , & if\ i = j \\ -1\ , & if\ \exists\ e(i,j) \\ 0\ , & otherwise \end{cases}$$

- Laplacian matrix 5x5:

$$\mathbf{L} = \begin{pmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{pmatrix}$$



Graph cuts

- Graph Laplacian: $\mathbf{L} = \mathbf{D} - \mathbf{A}$
- Graph cut:

$$Q(\mathbf{s}) = cut(V^+, V^-) = \frac{1}{4}\sum_{i,j} L_{ij} s(i) s(j) = \frac{\mathbf{s}^T \mathbf{L} \mathbf{s}}{4}$$

- Minimal cut:

$$\min_{\mathbf{s}} Q(\mathbf{s})$$

- Balanced cut constraint:

$$\sum_i s(i) = 0$$

- Integer minimization problem, exact solution is NP-hard!

Spectral method - relaxation

- Discrete problem $\rightarrow$ continuous problem
- Discrete problem: find

$$\min_{\mathbf{s}}(\frac{1}{4}\mathbf{s}^T\mathbf{L}\mathbf{s})$$

under constraints: $s(i) = \pm 1$, $\sum_i s(i) = 0$;

- Relaxation - continuous problem: find

$$\min_{\mathbf{x}}(\frac{1}{4}\mathbf{x}^T\mathbf{L}\mathbf{x})$$

under constraints: $\sum_i x(i)^2 = n$, $\sum_i x(i) = 0$

- Given $x(i)$, round them up by $s(i) = sign(x(i))$
- Exact constraint satisfies relaxed equation, but not other way around!

Spectral method - computations

- Constraint optimization problem (Lagrange multipliers):

$$Q(\mathbf{x}) = \frac{1}{4}\mathbf{x}^T\mathbf{L}\mathbf{x} - \lambda(\mathbf{x}^T\mathbf{x} - n), \;\; \mathbf{x}^T\mathbf{e} = 0$$

- Eigenvalue problem:

$$\mathbf{L}\mathbf{x} = \lambda\mathbf{x}, \;\; \mathbf{x} \perp \mathbf{e}$$

- Solution:

$$Q(\mathbf{x_i}) = \frac{n}{4}\lambda_i$$

- First (smallest) eigenvector:

$$\mathbf{L}\mathbf{e} = 0, \;\; \lambda = 0, \;\; \mathbf{x_1} = \mathbf{e}$$

- Looking for the second smallest eigenvalue/eigenvector λ_2 and $\mathbf{x}_2$
- Minimization of Rayleigh-Ritz quotient:

$$\min_{\mathbf{x} \perp \mathbf{x}_1} \left(\frac{\mathbf{x}^T\mathbf{L}\mathbf{x}}{\mathbf{x}^T\mathbf{x}} \right)$$

Spectral graph partitioning algorithm

Algorithm: Spectral graph partitioning - normalized cuts

Input: adjacency matrix $\mathbf{A}$

Output: class indicator vector $\mathbf{s}$

compute $\mathbf{D} = diag(deg(\mathbf{A}))$;

compute $\mathbf{L} = \mathbf{D} - \mathbf{A}$;

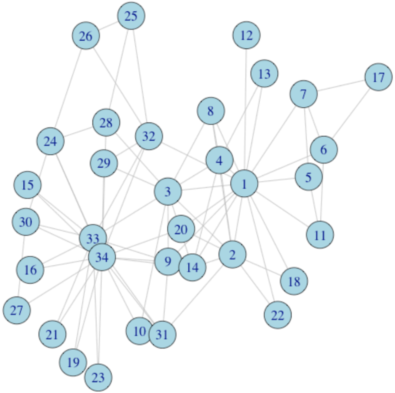
solve for second smallest eigenvector:

min cut: $\mathbf{Lx} = \lambda\mathbf{x}$;

normalized cut : $\mathbf{Lx} = \lambda\mathbf{Dx}$;

set $\mathbf{s} = sign(\mathbf{x}_2)$

Example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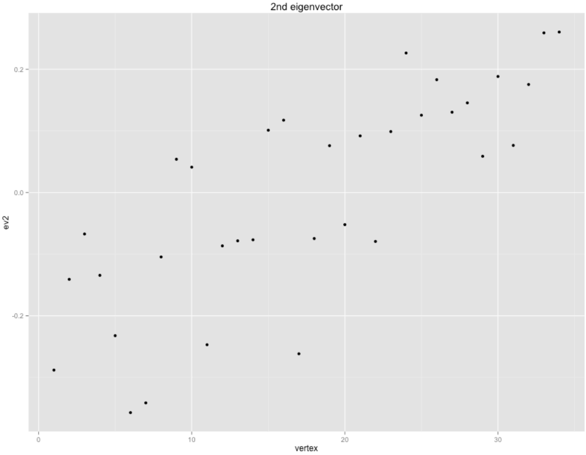
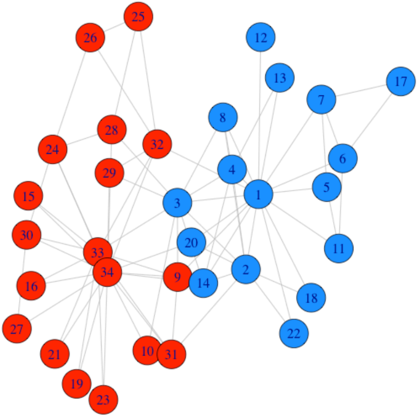
Example



Eigenvalues: $\lambda_1 = 0$, $\lambda_2 = 0.2$, $\lambda_3 = 0.25$ ...

Example



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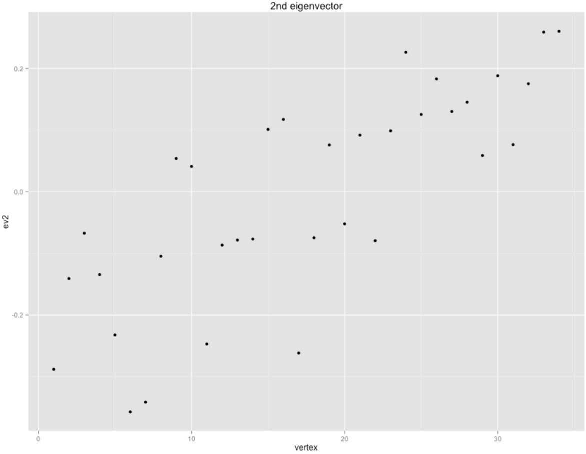
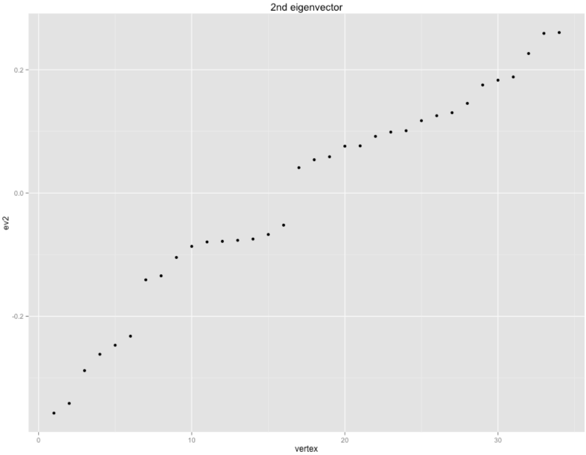
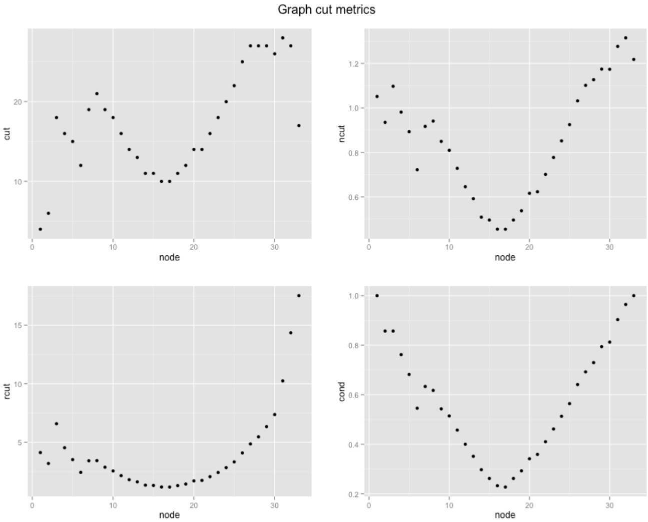
Example



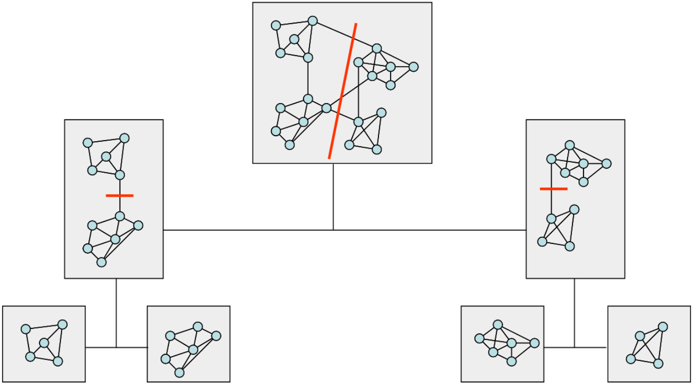
Spectral ordering



Cut metrics

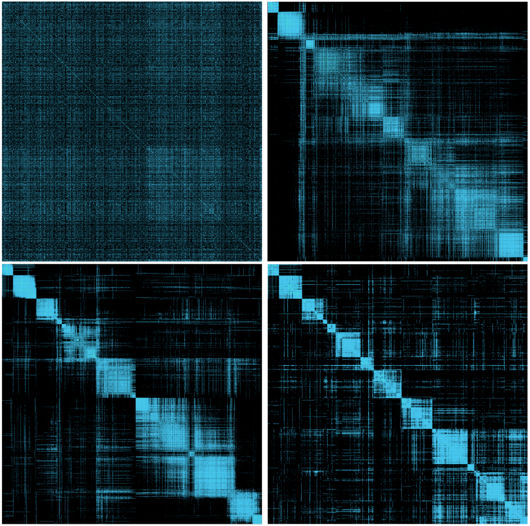


Multilevel spectral



recursive partitioning

Multilevel spectral



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