NATIONAL RESEARCH UNIVERSITY

Structural properties of networks

Network Science
Lecture 6

Leonid Zhukov

lzhukov@hse.ru

www.leonidzhukov.net/hse/2022/networks

National Research University Higher School of Economics
School of Data Analysis and Artificial Intelligence, Department of Computer Science

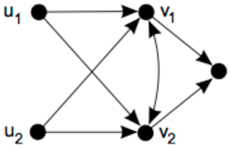
Patterns of relations

- Global, statistical properties of the networks:
  - average node degree (degree distribution)
  - average clustering
  - average path length
- Local, per vertex properties:
  - node centrality
  - page rank
- Pairwise properties:
  - node equivalence
  - node similarity
  - correlation between pairs of vertices (node values)

Structural equivalence

Definition

Structural equivalence: two vertices are structurally equivalent if their respective sets of in-neighbors and out-neighbors are the same

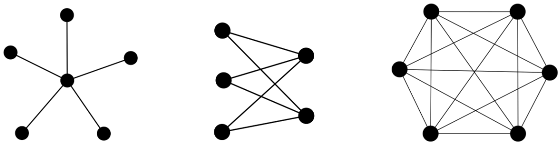


	u1	u2	v1	v2	w
u1	0	0	1	1	0
u2	0	0	1	1	0
v1	0	0	0	1	1
v2	0	0	1	0	1
w	0	0	0	0	0

rows and columns of adjacency matrix of structurally equivalent nodes are identical, ”connect to the same neighbors”

Structural equivalence

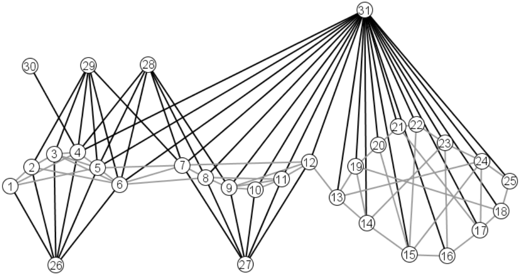
- In order for adjacent vertices to be structurally equivalent, they should have self loops.
- Sometimes called ”strong structural equivalence”
- Sometimes relax requirements for self loops for adjacent nodes



Structural similarity

Definition

Two nodes are similar to each other if they share many neighbors.



Similarity measures

- Jaccard similarity

$$J(v_i, v_j) = \frac{|\mathcal{N}(v_i) \cap \mathcal{N}(v_j)|}{|\mathcal{N}(v_i) \cup \mathcal{N}(v_j)|}$$

- Cosine similarity (vectors in *n*-dim space)

$$\sigma(v_i, v_j) = cos(\theta_{ij}) = \frac{\mathbf{v}_i^T \mathbf{v}_j}{|\mathbf{v}_i||\mathbf{v}_j|} = \frac{\sum_k A_{ik} A_{kj}}{\sqrt{\sum A_{ik}^2} \sqrt{\sum A_{jk}^2}}$$

- Pearson correlation coefficient:

$$r_{ij} = \frac{\sum_k (A_{ik} - \langle A_i \rangle)(A_{jk} - \langle A_j \rangle)}{\sqrt{\sum_k (A_{ik} - \langle A_i \rangle)^2} \sqrt{\sum_k (A_{jk} - \langle A_j \rangle)^2}}$$

Similarity measures

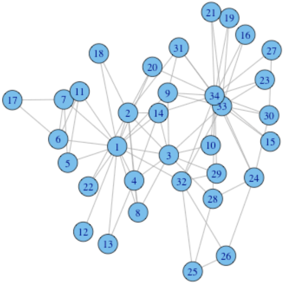
- Unweighted undirected graph $A_{ik} = A_{ki}$, binary matrix, only 0 and 1
- $\sum_k A_{ik} = \sum_k A_{ik}^2 = k_i$ - node degree
- $\sum_k A_{ik}A_{kj} = (A^2)_{ij} = n_{ij}$ - number of shared neighbors
- Cosine similarity (vectors in *n*-dim space)

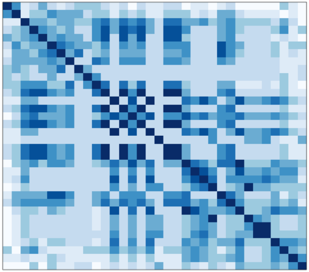
$$\sigma(v_i, v_j) = cos(\theta_{ij}) = \frac{n_{ij}}{\sqrt{k_i k_j}}$$

- Pearson correlation coefficient:

$$r_{ij} = \frac{n_{ij} - \frac{k_i k_j}{n}}{\sqrt{k_i - \frac{k_i^2}{n}}\sqrt{k_j - \frac{k_j^2}{n}}}$$

Similarity matrix





Graph

Node similarity matrix

SimRank

- G - directed graph
- Two vertices are similar if they are referenced by similar vertices
- $s(a, b)$ - similarity between a and b, $I()$ - set of in-neighbours

$$s(a,b) = \frac{C}{|I(a)||I(b)|} \sum_{i=1}^{I(a)} \sum_{j=1}^{I(b)} s(I_i(a), I_j(b)), \;\; a \neq b$$

$$s(a,a) = 1$$

- Matrix notation:

$$S_{ij} = \frac{C}{k_i k_j} \sum_{k,m} A_{ki} A_{mj} S_{km}$$

- Iterative solution starting from $s_0(i,j) = \delta_{ij}$

Jeh and Widom, 2002

Degree correlation

Degree correlation is the likelyhood that nodes link to nodes with similar or dissimilar nodal degree.

- Pearson degree correlation coefficient $(-1 \leq r \leq 1)$

$$r = \frac{cov}{var} = \frac{\sum_{ij} A_{ij}(k_i - \langle k \rangle)(k_j - \langle k \rangle)}{\sum_{ij} A_{ij}(k_i - \langle k \rangle)^2}$$

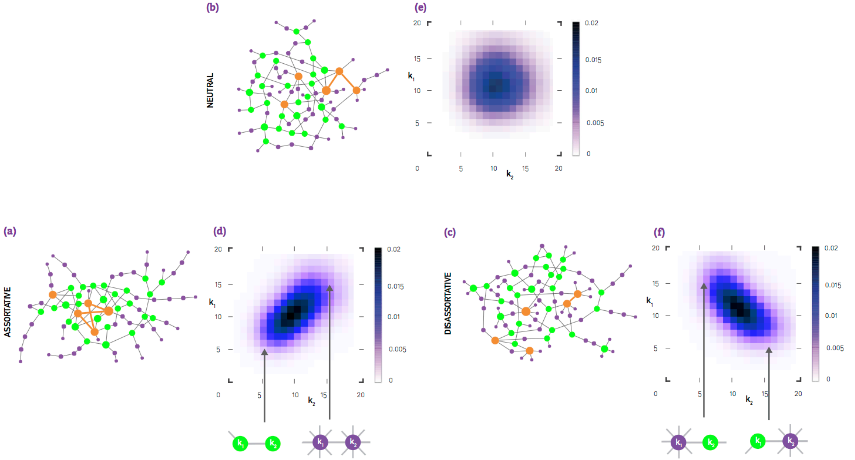
- Degree correlation matrix - fraction of edges connecting nodes degrees k, k'

$$e_{k,k'} = \frac{m(k,k')}{m}$$

- Degree correlation function

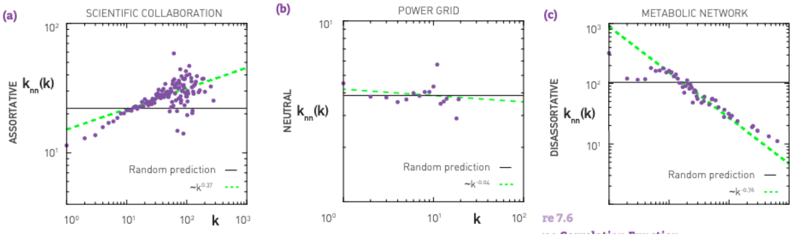
$$k_{nn}(k) = \sum_{k'} k' P(k'|k); \qquad P(k'|k) = \frac{e_{k,k'}}{\sum_k' e_{k,k'}}$$

Degree correlation matrix



from A.L. Barabasi, 2016

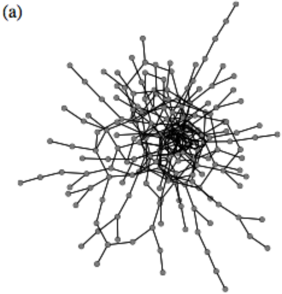
Degree correlation function

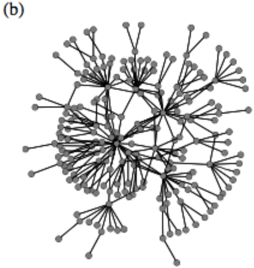


from A.L. Barabasi, 2016

Assortative and diassortative networks

- Assortative network (r > 0): hubs (high degree nodes) tend to connect to hubs, low degree nodes to low degree nodes
- Disassortative network (r <0): high degree nodes connected to low degree nodes, star-like structure





Assortative network $(r > 0)$ Disassortative network $(r < 0)$

Mixing patterns in networks

Network mixing patterns

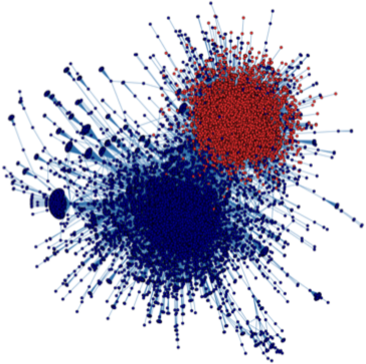
- **Assortative mixing**, "like links with like", attributed of connected nodes tend to be more similar than if there were no such edge
- **Disassortative mixing**, "like links with dislike", attributed of connected nodes tend to be less similar than if there were no such edge

Vertices can mix on any vertex attributes (age, sex, geography in social networks), unobserved attributes, vertex degrees

Examples:
assortative mixing - in social networks political beliefs, obesity, race
disassortative mixing - dating network, food web (predator/prey), economic networks (producers/consumers)

Assortative mixing

- Political polarization on Twitter: political retweet network ,red color - ”right-learning” users, blue color - ”left learning” users



- Assortative mixing = homophily

Conover et al., 2011

Mixing by categorical attributes

- Vertex categorical attribute (c_i -label): color, gender, ethnicity
- How much more often do attributes match across edges than expected at random?
- Modularity :

$$Q = \frac{m_c - \langle m_c \rangle}{m} = \frac{1}{2m} \sum_{ij} \left(A_{ij} - \frac{k_i k_j}{2m} \right) \delta(c_i, c_j)$$

- m_c - number of edges between vertices with same attributes $\langle m_c \rangle$ - expected number of edges within the same class in random network
- Assortativity coefficient:

$$\frac{Q}{Q_{max}} = \frac{\sum_{ij} \left(A_{ij} - \frac{k_i k_j}{2m} \right) \delta(c_i, c_j)}{2m - \sum_{ij} \frac{k_i k_j}{2m} \delta(c_i, c_j)}$$

Mixing by scalar values

- Vertex scalar value (attribute) - x_i
- How much more similar are attributes across edges than expected at random?
- Average and covariance over edges

$$\begin{aligned}
\langle x \rangle &= \frac{\sum_i k_i x_i}{\sum_i k_i} = \frac{1}{2m} \sum_i k_i x_i = \frac{1}{2m} \sum_{ij} A_{ij} x_i \\
var &= \frac{1}{2m} \sum_{ij} A_{ij} (x_i - \langle x \rangle)^2 = \frac{1}{2m} \sum_i k_i (x_i - \langle x \rangle)^2 \\
cov &= \frac{1}{2m} \sum_{ij} A_{ij} (x_i - \langle x \rangle)(x_j - \langle x \rangle)
\end{aligned}$$

- Assortativity coefficient

$$r = \frac{cov}{var} = \frac{\sum_{ij} A_{ij} (x_i - \langle x \rangle)(x_j - \langle x \rangle)}{\sum_{ij} A_{ij} (x_i - \langle x \rangle)^2} = \frac{\sum_{ij} \left(A_{ij} - \frac{k_i k_j}{2m} \right) x_i x_j}{\sum_{ij} \left(k_i \delta_{ij} - \frac{k_i k_j}{2m} \right) x_i x_j}$$

Mixing by node degree

- Assortative mixing by node degree, $x_i \leftarrow k_i$

$$r = \frac{\sum_{ij}\left(A_{ij} - \frac{k_i k_j}{2m}\right) k_i k_j}{\sum_{ij}\left(k_i \delta_{ij} - \frac{k_i k_j}{2m}\right) k_i k_j}$$

- Computations:

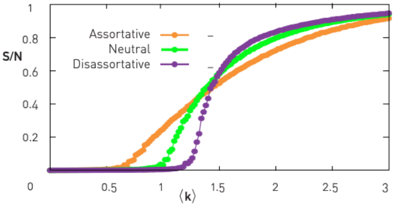
$$S_1 = \sum_i k_i = 2m$$
$$S_2 = \sum_i k_i^2$$
$$S_3 = \sum_i k_i^3$$
$$S_e = \sum_{ij} A_{ij} k_i k_j$$

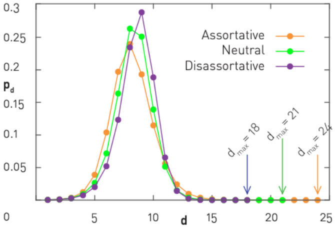
- Assortatitivity coefficient

$$r = \frac{S_e S_1 - S_2^2}{S_3 S_1 - S_2^2}$$

Random graph example

Random graph with $10,000$ nodes





Phase transition plot

Path length plot

from A.L. Barabasi, 2016

Friendship paradox

"On average, your friends are more popular, than you are"

- Average neighbour degree of a node with degree k

$$k_{nn} = \sum_{k'} k' P(k'|k)$$

- For uncorrelated network

$$k_{nn} = \sum_{k'} k' q'_k = \sum_{k'} k' \frac{k' p(k')}{\langle k \rangle} = \frac{\langle k^2 \rangle}{\langle k \rangle}$$

- in random network

$$\langle k^2 \rangle = \langle k \rangle (1 + \langle k \rangle)$$

- in scale free networks

$$\langle k^2 \rangle / \langle k \rangle \gg \langle k \rangle$$

We are more likely to be friends with hubs than with small-degree nodes, because hubs have more friends than the small nodes.

Typical network structure

Core-periphery structure of a network

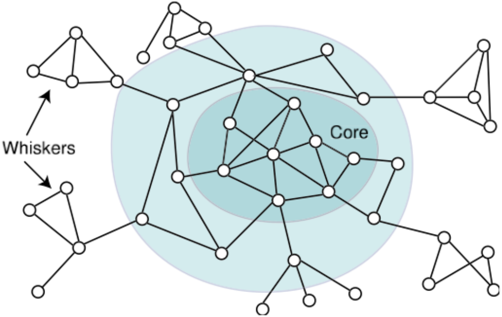
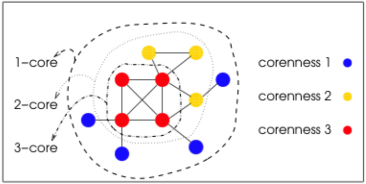


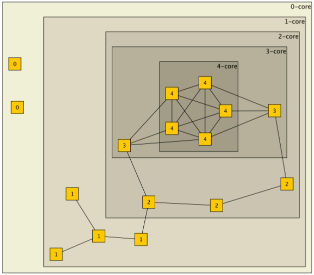
image from J. Leskovec, K. Lang, 2010

Graph cores

Definition

A *k-core* is the largest subgraph such that each vertex is connected to at least *k* others in subset





Every vertex in *k*-core has a degree $k_i \geq k$

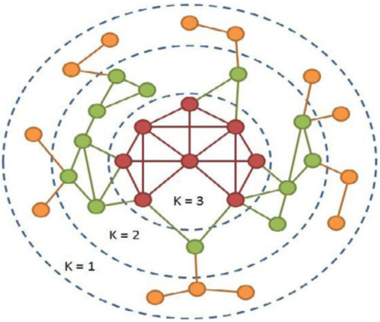
$(k+1)$-core is always subgraph of *k*-core

The core number of a vertex is the highest order of a core that contains this vertex

k-core decomposition

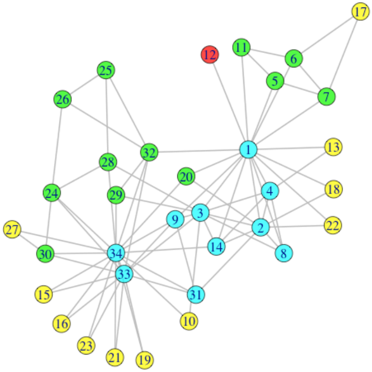
V. Batageli, M. Zaversnik, 2002

- If from a given graph G = (V, E) recursively delete all vertices, and lines incident with them, of degree less than k, the remaining graph is the k-core.

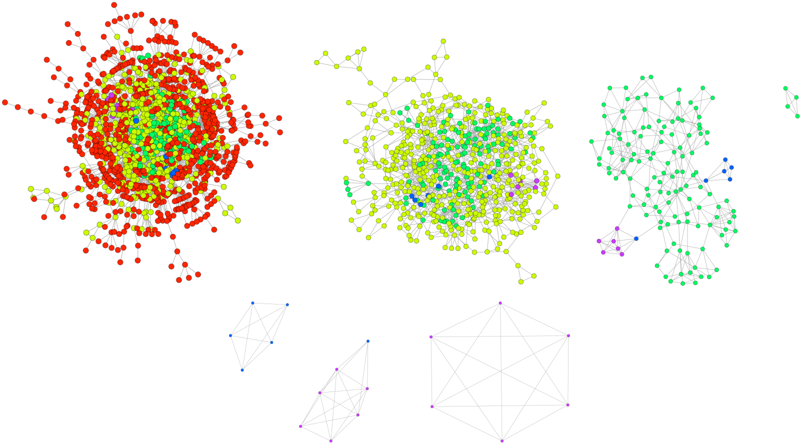


K-cores

Zachary karate club: 1,2,3,4 - cores



k-cores



k-cores: 1:1458, 2:594, 3:142, 4:12, 5:6

k-shells: 1:864-red, 2:452-pale green, 3:130-green, 5:6-blue, 6:6-purple

References

- White, D., Reitz, K.P. Measuring role distance: structural, regular and relational equivalence. Technical report, University of California, Irvine, 1985
- S. Borgatti, M. Everett. The class of all regular equivalences: algebraic structure and computations. Social Networks, v 11, p65-68, 1989
- E. A. Leicht, P.Holme, and M. E. J. Newman. Vertex similarity in networks. Phys. Rev. E 73, 026120, 2006
- G. Jeh and J. Widom. SimRank: A Measure of Structural-Context Similarity. Proceedings of the eighth ACM SIGKDD , p 538-543. ACM Press, 2002
- M. E. J. Newman. Assortative mixing in networks. Phys. Rev. Lett. 89, 208701, 2002.
- M. Newman. Mixing patterns in networks. Phys. Rev. E, Vol. 67, p 026126, 2003
- M. McPherson, L. Smith-Lovin, and J. Cook. Birds of a Feather: Homophily in Social Networks, Annu. Rev. Sociol, 27:415-44, 2001.