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A Necessary and Sufficient Condition for Reaching a Consensus Using DeGroot's Method

ROGER L. BERGER*

DeGroot (1974) proposed a model in which a group of k individuals might reach a consensus on a common subjective probability distribution for an unknown parameter. This paper presents a necessary and sufficient condition under which a consensus will be reached by using DeGroot's method. This work corrects an incorrect statement in the original paper about the conditions needed for a consensus to be reached. The condition for a consensus to be reached is straightforward to check and yields the value of the consensus, if one is reached.

KEY WORDS: Subjective probability distribution; Markov chain; Stochastic matrix; Opinion pool.

1. INTRODUCTION

Consider a group of k individuals, each of whom can specify his or her own subjective probability distribution for the unknown value of some parameter θ. Suppose the k individuals must act together as a team or committee. DeGroot (1974) presented a model in which the group might reach a consensus and form a common subjective probability distribution for θ by pooling their opinions. DeGroot's method is both simple and intuitively appealing. For this reason, it has been cited by many authors, including Aumann (1976), Dickey and Freeman (1975), Dickey and Gunel (1978), Hogarth (1975), Moskowitz, Schaefer, and Borcherding (1976), Ng (1977), Press (1978), and Woodworth (1976).

This paper presents a necessary and sufficient condition under which a consensus will be reached by using DeGroot's method. DeGroot presented one such condition, but that condition turns out to be sufficient but not necessary. So this paper presents a weaker condition under which a consensus will be reached. The condition that must be checked to determine if a consensus can be reached is explicitly calculated. Roughly speaking, the result is that the group of k individuals can be partitioned into subgroups. The behavior of each subgroup determines whether the whole group will reach a consensus.

2. MODEL FOR REACHING A CONSENSUS

DeGroot (1974) presented the following model under which a consensus might be reached among the k individuals. A more detailed explanation of the model can be found in DeGroot's paper.

For $i = 1, \ldots, k$, let F_i denote the subjective probability distribution that individual i assigns to the parameter θ. The subjective distributions, $F_1, \ldots, F_k$, will be based on the different backgrounds and different levels of expertise of the members of the group. It is assumed that if individual i is informed of the distributions of each of the other members of the group, he might wish to revise his subjective distribution to accommodate this information. It is further assumed that when individual i makes this revision, his revised distribution is a linear combination of the distributions $F_1, \ldots, F_k$. Let p_{ij} denote the weight that individual i assigns to F_j when he makes this revision. It is assumed that the p_{ij}'s are all nonnegative and $\sum_{j=1}^{k} p_{ij} = 1$. So, after being informed of the subjective distributions of the other members of the group, individual i revises his own subjective distribution from F_i to $F_{i1} = \sum_{j=1}^{k} p_{ij}F_j$.

Let $\mathbf{P}$ denote the $k \times k$ matrix whose (i, j)th element is $p_{ij}(i = 1, \ldots, k; j = 1, \ldots, k)$. $\mathbf{P}$ is a stochastic matrix since the elements are all nonnegative and the rows sum to one. Let $\mathbf{F}$ and $\mathbf{F}^{(1)}$ be the vectors whose transposes are $\mathbf{F}' = (F_1, \ldots, F_k)$ and $\mathbf{F}^{(1)\prime} = (F_{11}, \ldots, F_{k1})$. Then the vector of revised subjective distributions can be written as $\mathbf{F}^{(1)} = \mathbf{PF}$.

The critical step in this process is that now the above revision is iterated. It is assumed that after individual i is informed of the revised subjective distributions, $F_{11}, \ldots, F_{k1}$, of the members of the group, he revises his subjective distribution from F_{i1} to $F_{i2} = \sum_{j=1}^{k} p_{ij}F_{j1}$. The process continues in this way. Let F_{in} denote the subjective distribution of individual i after n revisions. Let $\mathbf{F}^{(n)}$ denote the vector whose transpose is $F^{(n)\prime} = (F_{1n}, \ldots, F_{kn})$. Then $\mathbf{F}^{(n)} = \mathbf{PF}^{(n-1)} = \mathbf{P}^n\mathbf{F}$, $n = 2, 3, \ldots$. It is assumed that these revisions are made indefinitely or until $\mathbf{F}^{(n+1)} = \mathbf{F}^{(n)}$ for some n.

DeGroot states that *a consensus is reached* if and only if all k components of $\mathbf{F}^{(n)}$ converge to the same limit as $n \to \infty$. That is to say, a consensus is reached if and only if there exists a distribution F^* such that $\lim_{n\to\infty} F_{in} = F^*$, $i = 1, \ldots, k$.

DeGroot goes on to assert that a consensus is reached if and only if every row of the matrix $\mathbf{P}^n$ converges to the same vector, say $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_k)$. This is clearly a sufficient condition for a consensus to be reached. But it is *not* a necessary condition, as can be seen from this

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simple example. Suppose $F_1 = F_2 = \cdots = F_k$. Then it makes no difference what $\mathbf{P}$ is, since $\mathbf{F}^{(n)} = \mathbf{P}^n\mathbf{F} = \mathbf{F}$, $n = 2, 3, \ldots$. Thus the consensus F_1 is reached no matter what weights p_{ij} are used.

Whether a consensus is reached depends not only on $\mathbf{P}$ (as suggested by DeGroot's condition) but also on $\mathbf{F}$. The remainder of this paper explains how to check if a consensus is reached and how to calculate the consensus if one is reached for an arbitrary set of weights $\mathbf{P}$ and an arbitrary set of initial subjective distributions $\mathbf{F}$.

Chatterjee and Seneta (1977) consider a generalization of DeGroot's model in which the individuals can change their weights p_{ij} at each iteration. They consider conditions under which a consensus will be reached using this more general model. But they only consider the situation in which all the rows of the weight matrix converge to a common vector. So they do not take into account the effect of $\mathbf{F}$ on whether a consensus is reached.

3. CONDITION FOR CONVERGENCE

Since the matrix $\mathbf{P}$ is a $k \times k$ stochastic matrix, it can be regarded as the one-step transition probability matrix of a Markov chain with k states and stationary transition probabilities. With this interpretation, standard results about Markov chains can be applied here. These results will be used freely in this discussion. Standard references such as Chung (1960) and Karlin (1969) may be consulted for statements of these results.

DeGroot showed that if the Markov chain is irreducible and aperiodic, then a consensus will always be reached. If the chain is reducible or periodic, it is possible to reach a consensus if the subjective distributions F_i satisfy certain conditions. To state and check these conditions, it is necessary to partition the chain into its recurrent classes and moving subclasses of periodic classes. The following notation makes this partitioning explicit.

By appropriately relabeling the individuals in the group, the matrix $\mathbf{P}$ can be put into this form:

$$\mathbf{P} = \begin{pmatrix} \mathbf{P}_1 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_2 & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{P}_m & \mathbf{0} \\ \hline & & \mathbf{P}_{m+1} & & \end{pmatrix}.$$

Here $\mathbf{P}_i$ is an $m_i \times m_i$ matrix, $i = 1, \ldots, m$. $\mathbf{P}_{m+1}$ is an $m_{m+1} \times k$ matrix. In this Markov chain there are m recurrent classes of communicating states. States 1 through m_1 form the first recurrent class, and all these states communicate with each other. States $m_1 + 1$ through $m_1 + m_2$ form the second recurrent class of communicating states, and so on. States $(\sum_{i=1}^m m_i) + 1$ through k are the transient states. If there are no transient states in the chain, m_{m+1} is taken to be zero and $\mathbf{P}_{m+1}$ is not in the matrix.

Let d_i denote the period of the ith recurrent class. If the class is aperiodic, $d_i = 1$. Then by appropriately relabeling the individuals in the class, $\mathbf{P}_i$ can be written in the form:

$$\mathbf{P}_i = \begin{pmatrix} \mathbf{0} & \mathbf{P}_{i1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{P}_{i2} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{P}_{id_i-1} \\ \mathbf{P}_{id_i} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{pmatrix}.$$

Here $\mathbf{P}_{ij}$ is an $m_{ij} \times m_{i(j+1)}$ matrix, $j = 1, \ldots, d_i$. All of the m_{ij} are positive integers, $m_{i1} = m_{i(d_i+1)}$, and $\sum_{j=1}^{d_i} m_{ij} = m_i$. If the class is aperiodic, let $\mathbf{P}_{i1} = \mathbf{P}_i$ and interpret the above notation as $\mathbf{P}_i = \mathbf{P}_{i1}$. Let $M_1 = 0$ and $M_i = \sum_{j=1}^{i-1} m_j$, $i = 2, \ldots, m$. The states $M_i + 1$ through $M_i + m_{i1}$ are called the first moving subclass of the ith recurrent class. The states $M_i + m_{i1} + 1$ through $M_i + m_{i1} + m_{i2}$ are called the second moving subclass of the ith recurrent class, and so on.

Then all of the recurrent states in the chain (and hence all of the individuals in the group corresponding to these recurrent states) can be partitioned into subgroups according to which moving subclass they belong to. There are $d = \sum_{i=1}^m d_i$ subgroups in this partition.

For $i = 1, \ldots, m$ and $j = 1, \ldots, d_i$, let $\mathbf{A}_{ij}$ denote the $m_{ij} \times m_{ij}$ matrix given by $\mathbf{A}_{ij} = \mathbf{P}_{ij}\mathbf{P}_{i(j+1)} \cdots \mathbf{P}_{id_i}\mathbf{P}_{i1} \cdots \mathbf{P}_{i(j-1)}$.

Then $\mathbf{P}_i^{d_i}$ is given by

$$\mathbf{P}_i^{d_i} = \begin{pmatrix} \mathbf{A}_{i1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{i2} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{A}_{id_i} \end{pmatrix}.$$

Let $\boldsymbol{\pi}(i, j) = (\pi(i, j)_1, \ldots, \pi(i, j)_{m_{ij}})$ be the solution to the linear equations $\boldsymbol{\pi}(i, j)\mathbf{A}_{ij} = \boldsymbol{\pi}(i, j)$ together with the equation $\sum_{l=1}^{m_{ij}} \pi(i, j)_l = 1$. Since $\mathbf{A}_{ij}$ is the one-step transition probability matrix for an irreducible aperiodic Markov chain, a solution $\boldsymbol{\pi}(i, j)$ exists and it is unique. Let $\mathbf{F}(i, j)$ denote the $m_{ij} \times 1$ vector of initial subjective probability distributions for the individuals in the jth moving subclass of the ith recurrent class. That is, $\mathbf{F}(i, j)$ is the vector whose transpose is $\mathbf{F}'(i, j) = (F_{M_{ij}+1}, \ldots, F_{M_{ij}+m_{ij}})$ where $M_{ij} = (\sum_{l=1}^{i-1} m_l) + (\sum_{l=1}^{j-1} m_{il})$ and any sum from one to zero is defined to be zero.

Now the necessary and sufficient condition for a consensus to be reached can be stated. Theorem 1 gives the limiting distribution for a recurrent individual if such a limit exists. Theorem 2 gives the necessary and sufficient condition for the group to reach a consensus. The proofs of both theorems are given in Section 6.

Theorem 1. If individual l is in the jth moving subclass of the ith recurrent class and if $\lim_{n\to\infty} F_{ln}$ exists, then $\lim_{n\to\infty} F_{ln} = \boldsymbol{\pi}(i, j)\mathbf{F}(i, j)$.

Theorem 2. (a) If $d = 1$, a consensus is reached and the consensus is $\boldsymbol{\pi}(1, 1)\mathbf{F}(1, 1)$.

(b) If $d > 1$, a consensus is reached if and only if $\boldsymbol{\pi}(i, j)\mathbf{F}(i, j) = F^*$ for every $i = 1, \ldots, m$; $j = 1, \ldots, d_i$, for some distribution F^*. The consensus, if it is reached, is F^*.

Case (a), $d = 1$, is the case considered by DeGroot for, in this situation, all of the rows of $\mathbf{P}^n$ converge to the vector $(\boldsymbol{\pi}(1, 1)\mathbf{0})$ where $\mathbf{0}$ is a $1 \times m_2$ vector of zeros and m_2 is the number of transient states. But case (b), $d > 1$, gives the condition under which a consensus will be reached in the situation in which DeGroot claimed that a consensus would not be reached, namely, if there are at least two disjoint classes of communicating states or at least one class of communicating states is periodic.

The conditions under which a consensus will be reached in the reducible or periodic case may be seldom satisfied in practice. If many of the F_i's are equal, then the condition may be satisfied. But since the individuals in one recurrent class give zero weight to the opinions of the individuals in another recurrent class, it is hard to imagine that a particular linear combination of opinions from the first class would equal a particular linear combination of opinions from the second class. Nevertheless, the result is interesting in that it points out this fact. Reaching a consensus is determined not only by the general opinions of the individuals about one another, as expressed by $\mathbf{P}$, but also by the specific opinions of the individuals about the problem at hand, as expressed by $\mathbf{F}$. This fact would probably be true for any reasonable opinion pooling scheme.

4. AN EXAMPLE

The notation of Section 3 and the results of Theorems 1 and 2 will be illustrated with the following example. Suppose $k = 8$ and

$$\mathbf{P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} \end{pmatrix}.$$

Then $m = 2$, $d_1 = 1$, $d_2 = 2$, and $d = d_1 + d_2 = 3$.

$$\mathbf{P}_1 = \mathbf{P}_{11} = \mathbf{A}_{11} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

and $\boldsymbol{\pi}(1, 1)$, the solution to $\boldsymbol{\pi}(1, 1)\mathbf{A}_{11} = \boldsymbol{\pi}(1, 1)$ and $\sum_{l=1}^{3} \pi(1, 1)_l = 1$, is $(\frac{4}{11}, \frac{3}{11}, \frac{4}{11})$.

$$\mathbf{P}_2 = \begin{pmatrix} \mathbf{0} & \mathbf{P}_{21} \\ \mathbf{P}_{22} & \mathbf{0} \end{pmatrix}$$

where

$$\mathbf{P}_{21} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix} \quad \text{and} \quad \mathbf{P}_{22} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

$$\mathbf{A}_{21} = \begin{pmatrix} \frac{5}{12} & \frac{7}{12} \\ \frac{11}{24} & \frac{13}{24} \end{pmatrix} \quad \text{and} \quad \mathbf{A}_{22} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{3}{8} & \frac{5}{8} \end{pmatrix}.$$

Solving the linear equations yields $\boldsymbol{\pi}(2, 1) = (11/25, 14/25)$ and $\boldsymbol{\pi}(2, 2) = (9/25, 16/25)$. Theorem 2 states that a consensus is reached if and only if

$$\tfrac{4}{11} F_1 + \tfrac{3}{11}F_2 + \tfrac{4}{11}F_3 = \tfrac{11}{25}F_4 + \tfrac{14}{25}F_5 = \tfrac{9}{25}F_6 + \tfrac{16}{25}F_7.$$

The consensus, if it is reached, is the common value. In this example, the eighth state is transient and has no effect on whether a consensus is reached. Also, F_8 does not enter into the calculation of the consensus.

5. A COMPUTATIONAL SHORTCUT

To determine if a consensus is reached, it is necessary to compute the vectors $\boldsymbol{\pi}(i, j)$ ($i = 1, \ldots, m$; $j = 1, \ldots, d_i$). Each of these vectors is defined as the solution of a certain set of linear equations. The following result states that for each $i = 1, \ldots, m$, it is only necessary to solve the linear equations for $\boldsymbol{\pi}(i, 1)$. The remaining $d_i - 1$ vectors, $\boldsymbol{\pi}(i, 2), \ldots, \boldsymbol{\pi}(i, d_i)$, can be determined by simple matrix multiplication.

Theorem 3. For any $i = 1, \ldots, m$ and $j = 2, \ldots, d_i$, $\boldsymbol{\pi}(i, j) = \boldsymbol{\pi}(i, j - 1)\mathbf{P}_{i(j-1)}$.

Remark. For example, in the previous example it is easily verified that $\boldsymbol{\pi}(2, 2) = (9/25, 16/25) = \boldsymbol{\pi}(2, 1)\mathbf{P}_{21}$.

Proof. It suffices to show that $\boldsymbol{\pi}(i, j - 1)\mathbf{P}_{i(j-1)}$ satisfies the appropriate linear equalities; that is, the sum of the coordinates of $\boldsymbol{\pi}(i, j - 1)\mathbf{P}_{i(j-1)}$ is one and $\boldsymbol{\pi}(i, j - 1)\mathbf{P}_{i(j-1)}\mathbf{A}_{ij} = \boldsymbol{\pi}(i, j - 1)\mathbf{P}_{i(j-1)}$. The sum of the coordinates is one since the sum of the coordinates of $\boldsymbol{\pi}(i, j - 1)$ is one and the sum of each row of $\mathbf{P}_{i(j-1)}$ is one. The definition of $\mathbf{A}_{i(j-1)}$ and $\mathbf{A}_{ij}$ and the fact that $\boldsymbol{\pi}(i, j - 1)\mathbf{A}_{i(j-1)} = \boldsymbol{\pi}(i, j - 1)$ yields

$$\begin{aligned} \boldsymbol{\pi}(i, j - 1)\mathbf{P}_{i(j-1)}\mathbf{A}_{ij} &= \boldsymbol{\pi}(i, j - 1)\mathbf{P}_{i(j-1)} \\ &\quad \times (\mathbf{P}_{ij} \ldots \mathbf{P}_{id_i}\mathbf{P}_{i1} \ldots \mathbf{P}_{i(j-1)}) \\ &= \boldsymbol{\pi}(i, j - 1)\mathbf{A}_{i(j-1)}\mathbf{P}_{i(j-1)} \\ &= \boldsymbol{\pi}(i, j - 1)\mathbf{P}_{i(j-1)}. \end{aligned}$$

Hence the second equality is also true.

6. PROOFS OF THEOREMS 1 AND 2

Let $\mathbf{p}_i^{(n)}$ denote the ith row of $\mathbf{P}^n$, $i = 1, \ldots, k$. Let $\mathbf{0}_j$ denote a $1 \times j$ vector of zeros. All of the limiting results for stochastic matrices used in these two proofs are summarized in Part I, Section 6, Theorem 4 of Chung (1960).

Proof of Theorem 1. Suppose l is in the jth moving subclass of the ith recurrent class. Then $\lim_{n\to\infty} \mathbf{p}_l^{(nd_i)}$ exists and is equal to $p_l^* = (\mathbf{0}_{M_{ij}}\boldsymbol{\pi}(i, j)\mathbf{0}_{k-M_{ij}-m_{ij}})$. So $\lim_{n\to\infty} F_{l(nd_i)} = \lim_{n\to\infty} \mathbf{p}_l^{(nd_i)}\mathbf{F} = \mathbf{p}_l^*\mathbf{F} = \boldsymbol{\pi}(i, j)\mathbf{F}(i, j)$. If $\lim_{n\to\infty} F_{ln}$ exists, it must equal the limit of the subsequence $F_{l(nd_i)}$. Therefore, $\lim_{n\to\infty} F_{ln} = \boldsymbol{\pi}(i, j)\mathbf{F}(i, j)$.

Proof of Theorem 2. (a) If $d = 1$, then there is only one recurrent class and it is aperiodic. So $\lim_{n\to\infty} \mathbf{p}_i^{(n)}$ exists and equals $\mathbf{p}^* = (\boldsymbol{\pi}(1, 1)\mathbf{0}_{m_2})$ for every $i = 1, \ldots, k$. Thus $\lim_{n\to\infty} F_{in} = \lim_{n\to\infty} \mathbf{p}_i^{(n)}\mathbf{F} = \mathbf{p}^*\mathbf{F} = \boldsymbol{\pi}(1, 1)\mathbf{F}(1, 1)$ for every $i = 1, \ldots, k$. So a consensus is reached and the consensus is $\boldsymbol{\pi}(1, 1)\mathbf{F}(1, 1)$.

(b) (Necessity). Suppose a consensus is reached. Then $\lim_{n\to\infty} F_{in} = F^*$ for every $i = 1, \ldots, k$. If l is in the jth moving class of the ith recurrent class, by Theorem 1, $\boldsymbol{\pi}(i, j)\mathbf{F}(i, j) = \lim_{n\to\infty} F_{ln} = F^*$. Thus $\boldsymbol{\pi}(i, j)\mathbf{F}(i, j) = F^*(i = 1, \ldots, m; j = 1, \ldots, d_i)$.

(b) (Sufficiency). Suppose $\boldsymbol{\pi}(i, j)\mathbf{F}(i, j) = F^*(i = 1, \ldots, m, j = 1, \ldots, d_i)$.

First, it will be shown that if l is a recurrent state, $\lim_{n\to\infty} F_{ln}$ exists and equals F^*. Suppose l is in the jth moving subclass of the ith recurrent class. Then, for $r = 0, \ldots, d_i - 1$, $\lim \mathbf{p}_l^{(nd_i+r)}$ exists and equals $\mathbf{p}_l^*(r) = (\mathbf{0}_{M_{iq}}\boldsymbol{\pi}(i, q)\mathbf{0}_{k-M_{iq}-m_{iq}})$ where $q = (j + r)(\text{mod } d_i)$. (Note that here $M_{i0} = M_{id_i}$, $m_{i0} = m_{id_i}$, $\boldsymbol{\pi}(i, 0) = \boldsymbol{\pi}(i, d_i)$, and $\mathbf{F}(i, 0) = \mathbf{F}(i, d_i)$ for $i = 1, \ldots, m$.) Thus $\lim_{n\to\infty} F_{l(nd_i+r)} = \lim_{n\to\infty} \mathbf{p}_l^{(nd_i+r)}\mathbf{F} = \mathbf{p}_l^*(r)\mathbf{F} = \boldsymbol{\pi}(i, q)\mathbf{F}(i, q) = F^*$. Since each of the d_i subsequences $F_{l(nd_i+r)}$, $r = 0, \ldots, d_i - 1$, converges to F^*, the full sequence F_{ln} also converges to F^*. Thus, since l was an arbitrary recurrent state, every subjective distribution corresponding to a recurrent state converges to F^*.

Finally, it will be shown that if l is a transient state, $\lim_{n\to\infty} F_{ln}$ exists and equals F^*. Let $\delta = \prod_{i=1}^{m} d_i$. Then, for $r = 0, \ldots, \delta - 1$, $\lim_{n\to\infty} \mathbf{p}_l^{(n\delta+r)}$ exists and equals $\mathbf{p}_l^*(r) = (f_{l11}^*(r)\boldsymbol{\pi}(1, 1), f_{l12}^*(r)\boldsymbol{\pi}(1, 2), \ldots, f_{lmd_m}^*(r)\boldsymbol{\pi}(m, \mathrm{d}_m), \mathbf{0}_{m_{m+1}})$ where $f_{lij}^*(r)$ is the probability that the chain is in the jth moving subclass of the ith recurrent class for some $n = r(\text{mod } d_i)$ given that the chain started in state l. (Note that the fact that the $f_{ij}^*(r)$, as defined by Chung, are constant for j in a particular moving subclass was used to express $\mathbf{p}_l^*(r)$ in terms of the $f_{lij}^*(r)$. Also note that $\sum_{i=1}^{m} \sum_{j=1}^{d_i} f_{lij}^*(r) = 1$.) Thus,

$$\begin{aligned}\lim_{n\to\infty} F_{l(n\delta+r)} &= \lim_{n\to\infty} \mathbf{p}_l^{(n\delta+r)}\mathbf{F} \\ &= \mathbf{p}_l^*(r)\mathbf{F} \\ &= \sum_{i=1}^{m} \sum_{j=1}^{d_i} f_{lij}^*(r)\boldsymbol{\pi}(i, j)\mathbf{F}(i, j) \\ &= \sum_{i=1}^{m} \sum_{j=1}^{d_i} f_{lij}^*(r)F^* \\ &= F^* \sum_{i=1}^{m} \sum_{j=1}^{d_i} f_{lij}^*(r) \\ &= F^*.\end{aligned}$$

Since each of the δ subsequences $F_{l(n\delta+r)}$, $r = 0, \ldots, \delta - 1$, converges to F^*, the full sequence F_{ln} also converges to F^*. Thus, since l was an arbitrary transient state, every subjective distribution corresponding to a transient state converges to F^*.

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