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Consider a group of individuals who must act together as a team or committee, and suppose that each individual in the group has his own subjective probability distribution for the unknown value of some parameter. A model is presented which describes how the group might reach agreement on a common subjective probability distribution for the parameter by pooling their individual opinions. The process leading to the consensus is explicitly described and the common distribution that is reached is explicitly determined. The model can also be applied to problems of reaching a consensus when the opinion of each member of the group is represented simply as a point estimate of the parameter rather than as a probability distribution.

1. INTRODUCTION

Consider a group of k individuals who must act together as a team or committee, and suppose that each of these k individuals can specify his own subjective probability distribution for the unknown value of some parameter θ . In this article we shall present a model which describes how the group might reach a consensus and form a common subjective probability distribution for θ simply by revealing their individual distributions to each other and pooling their opinions.

The problem of attaining agreement about subjective probability distributions has been discussed by many writers, including Eisenberg and Gale [4], Stone [13], Madansky [7], Norvig [9], Winkler [14], Morris [8], and Savage [11]. Good surveys of much of this previous work are given by Winkler [14], Staël von Holstein [12], and Bacharach [1].

For $i = 1, \dots, k$, we shall let F_i denote the subjective probability distribution which individual i assigns to the parameter θ . In the present work it is not necessary that θ be a real-valued parameter. Indeed, θ may be regarded as any arbitrary variable whose value is not completely known to the k individuals. The value of θ is assumed to lie in an abstract parameter space Ω that is endowed with a σ -field of measurable subsets for which probabilities can be specified. Thus, F_1, \dots, F_k are subjective probability distributions over Ω which represent the prior beliefs about θ of the k individuals. In other words, for any measurable set A in the parameter space Ω , $F_i(A)$ is the prior probability of individual i that the value of θ will lie in A .

If p_1, \dots, p_k are nonnegative constants such that $\sum_{i=1}^k p_i = 1$, then $\sum_{i=1}^k p_i F_i$ will denote the probability

distribution over Ω for which the probability of any measurable set A is $\sum_{i=1}^k p_i F_i(A)$. Some of the writers previously mentioned have suggested representing the overall opinion of the group by a probability distribution of the form $\sum_{i=1}^k p_i F_i$. Stone [13] has called such a linear combination an "opinion pool." The difficulty in using an opinion pool to represent the consensus of the group lies, of course, in choosing suitable weights p_1, \dots, p_k . In the model that will be presented in this article, the consensus that is reached by the group will have the form of an opinion pool. However, the model is new. It explicitly describes the process which leads to the consensus and explicitly specifies the weights that are to be used in the opinion pool.

In summary, this model is believed to have three important advantages:

1. The process that it describes is intuitively appealing.
2. It presents simple conditions for determining whether it is possible for the group to reach a consensus.
3. When a consensus can be reached, the weights to be used in this consensus can be explicitly and simply calculated.

2. WEIGHTING THE OPINIONS OF OTHERS

We shall now consider one of the individuals in the group and discuss how this individual might change his subjective distribution of θ when he is apprised of the subjective distributions of the others in the group. Because of the different backgrounds of the different members of the group, their subjective distributions F_1, \dots, F_k will typically have been developed from different types of information about θ and will typically reflect different levels of expertise among the members. Therefore, if individual i is apprised of the distribution F_j ($j \neq i$) of each of the other members of the group, it will be natural for him to revise his own subjective distribution F_i to accommodate the information and expertise, the opinions and judgments, of the rest of the group.

It is assumed that when individual i revises his subjective distribution in this way, his revised distribution will be a linear combination of the distributions F_1, \dots, F_k of the members of the group. For $i = 1, \dots, k$ and $j = 1, \dots, k$, we shall let p_{ij} denote the weight that individual i assigns to the distribution of individual j when he carries out this revision. It is assumed that

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$p_{ij} \geq 0$ for every value of i and j , and that $\sum_{j=1}^k p_{ij} = 1$ for every value of i . Thus, if individual i could learn the subjective distributions F_1, \dots, F_k of the other members of the group, then it is assumed that he would be willing to revise his own distribution from F_i to

$$F_{i1} = \sum_{j=1}^k p_{ij} F_j. \quad (2.1)$$

The weights p_{i1}, \dots, p_{ik} should be chosen by individual i , before he is informed of the distributions of the other members of the group, on the basis of the relative importance that he assigns to the opinions of the various members of the group, including himself. For example, if individual i feels that individual j is a leading expert with regard to predicting the value of the parameter θ or if he thinks that individual j has had access to a large amount of information about the value of θ , then individual i will choose a large value for p_{ij} . Alternatively, individual i may wish to assign a large weight p_{ii} to his own distribution F_i and small total weight to the distributions of the others. In this case, his revised subjective distribution F_{i1} will probably differ very little from F_i .

Next, we shall let \mathbf{P} denote the $k \times k$ matrix comprising the elements p_{ij} ($i = 1, \dots, k; j = 1, \dots, k$). It should be noted that \mathbf{P} is a stochastic matrix since each element p_{ij} is nonnegative and the sum of the elements in any given row is 1. Furthermore, if we let \mathbf{F} and $\mathbf{F}^{(1)}$ denote the column vectors whose transposes are

$$\mathbf{F}' = (F_1, \dots, F_k) \quad \text{and} \quad \mathbf{F}^{(1)'} = (F_{11}, \dots, F_{k1}), \quad (2.2)$$

then it follows from (2.1) that

$$\mathbf{F}^{(1)} = \mathbf{P}\mathbf{F}. \quad (2.3)$$

3. ITERATING THE PROCESS

We have seen that after each member of the group has been apprised of the distributions of the other members, the distributions of the k members will change from F_1, \dots, F_k to F_{11}, \dots, F_{k1} . We now come to the critical step of the process.

Individual i knows that his own subjective distribution has changed from F_i to F_{i1} . Suppose now that he is also informed that the other $k - 1$ members of the group have also changed their subjective distributions. If he still wishes to form his subjective distribution by assigning the weight p_{ij} to the distribution of individual j —and there does not seem to be any basis for his changing these weights at this time—then in order to remain consistent with this principle he must again revise his subjective distribution by forming the linear combination

$$F_{i2} = \sum_{j=1}^k p_{ij} F_{j1}. \quad (3.1)$$

In other words, the opinion of individual j has changed from F_j to F_{j1} ($j = 1, \dots, k$). Therefore, if individual i is going to revise his own subjective distribution by assigning the weight p_{ij} to that distribution, then his revised distribution F_{i2} will be given by (3.1).

The process continues in this way. Each revision by the members of their own distributions leads in turn to yet another revision as each member tries to update the linear combination he is using in order to take into account the latest changes of opinion of himself and of the others. Let F_{in} denote the distribution of individual i after n revisions ($i = 1, \dots, k; n = 1, 2, \dots$), and let $\mathbf{F}^{(n)}$ denote the $k \times 1$ column vector whose transpose is

$$\mathbf{F}^{(n)'} = (F_{1n}, \dots, F_{kn}). \quad (3.2)$$

Then it follows from (3.1) and (2.3) that $\mathbf{F}^{(2)} = \mathbf{P}\mathbf{F}^{(1)} = \mathbf{P}^2\mathbf{F}$, and in general, that

$$\mathbf{F}^{(n)} = \mathbf{P}\mathbf{F}^{(n-1)} = \mathbf{P}^n\mathbf{F}, \quad n = 2, 3, \dots \quad (3.3)$$

It is assumed that the members of the group continue to make these revisions indefinitely or until $\mathbf{F}^{(n+1)} = \mathbf{F}^{(n)}$ for some value of n , so that further revision does not actually change any member's subjective distribution.

4. CONVERGENCE TO A CONSENSUS

The subjective distributions of the k members of the group will converge to each other if and only if there is a distribution F^* such that

$$\lim_{n \rightarrow \infty} F_{in} = F^* \quad i = 1, \dots, k. \quad (4.1)$$

In other words we shall say that *a consensus is reached* if and only if all k components of $\mathbf{F}^{(n)}$ converge to the same limit as $n \rightarrow \infty$.

Now let $p_{ij}^{(n)}$ denote the element in row i and column j of the matrix $\mathbf{P}^{(n)}$. Then it follows from (3.3) that a consensus is reached if and only if there exists a vector $\boldsymbol{\pi} = (\pi_1, \dots, \pi_k)$ such that, for $i = 1, \dots, k$ and $j = 1, \dots, k$,

$$\lim_{n \rightarrow \infty} p_{ij}^{(n)} = \pi_j. \quad (4.2)$$

If (4.2) is satisfied for every value of i and j , then π_1, \dots, π_k are necessarily nonnegative and $\sum_{i=1}^k \pi_i = 1$. Thus, when a consensus is reached, the common subjective distribution of each of the k members of the group will be $\sum_{i=1}^k \pi_i F_i$.

5. CONDITIONS FOR CONVERGENCE

Since the matrix \mathbf{P} is a $k \times k$ stochastic matrix, it can be regarded as the one-step transition probability matrix of a Markov chain with k states and stationary transition probabilities. Because of this interpretation, the standard limit theorems of the theory of Markov chains can be applied here. The following theorem, which is adapted from a result given by Doob [3, p. 173], provides a simple condition for (4.2) to hold and, hence, for a consensus to be reached.

Theorem 1: If there exists a positive integer n such that every element in at least one column of the matrix \mathbf{P}^n is positive, then a consensus is reached.

An analogous result can be stated as follows:

Theorem 2: If all the recurrent states of the Markov chain communicate with each other and are aperiodic, then a consensus is reached.

On the other hand, if the states of the chain form at least two disjoint closed sets of communicating states or if the communicating states in a single closed set are periodic, then a consensus is not reached. These results are discussed in texts such as Feller [5], Karlin [6], or Parzen [10].

6. CALCULATION OF THE CONSENSUS

If in fact a consensus is reached, then the next result, which is also well known in the theory of Markov chains, indicates how the common limiting distribution which forms the consensus can be explicitly calculated. It should be recalled that a vector $\pi = (\pi_1, \dots, \pi_k)$ is said to be a *stationary probability vector* if $\pi\mathbf{P} = \pi$ and the components of π are nonnegative numbers whose sum is 1.

Theorem 3: Suppose that a consensus is reached and let $\sum_{i=1}^k \pi_i F_i$ denote the common subjective distribution that is reached in the consensus. Then $\pi = (\pi_1, \dots, \pi_k)$ is the unique stationary probability vector.

Thus, the values of π_1, \dots, π_k used in the consensus are calculated by solving the linear equations $\pi\mathbf{P} = \pi$ together with the equation $\sum_{i=1}^n \pi_i = 1$.

It is worthwhile noting that if some state i is a transient state and if a consensus is reached, then the value of π_i in the consensus will be $\pi_i = 0$. In other words, no weight will be assigned in the consensus to the prior distribution F_i of individual i .

6.1 Examples

1. Suppose first that there are just two individuals in the group, so that $k = 2$, and suppose that

$$\mathbf{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}.$$

In other words, individual 1 assigns equal weight to his own distribution and the distribution of individual 2, and individual 2 assigns three times as much weight to his own distribution as he does to the distribution of individual 1. It follows from Theorem 1 that a consensus is reached and it is easily found that the unique stationary probability vector is $(\frac{1}{3}, \frac{2}{3})$. Therefore, by Theorem 3, the distribution of both individuals in the consensus will be $(\frac{1}{3})F_1 + (\frac{2}{3})F_2$.

2. Suppose now that a third individual is added to the group and that the matrix \mathbf{P} is now expanded as follows:

$$\mathbf{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{3}{4} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

It again follows from Theorem 1 that a consensus is

reached, and the stationary probability vector is now found to be $(\frac{1}{3}, \frac{2}{3}, 0)$. Therefore, by Theorem 3, the common distribution of the three individuals in the consensus is again $(\frac{1}{3})F_1 + (\frac{2}{3})F_2$. Since state 3 is transient in this Markov chain, the prior distribution F_3 of individual 3 receives no weight in the consensus.

3. Suppose that in a group with just two individuals, we have

$$\mathbf{P} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Here, the two states do not communicate with each other and a consensus is not reached. More generally, suppose that in a group of four individuals, we have

$$\mathbf{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Now states 1 and 2 communicate with each other and individuals 1 and 2 will therefore reach a consensus between themselves. Their consensus will be $(\frac{1}{2})F_1 + (\frac{1}{2})F_2$, and it will be reached after only a single revision. Similarly, individuals 3 and 4 will reach a consensus between themselves, namely $(\frac{1}{2})F_3 + (\frac{1}{2})F_4$. But the entire group of four individuals will not reach a consensus.

In general any individuals for whom the corresponding states of the Markov chain form a closed communicating, aperiodic class will reach a consensus among themselves.

7. APPLICATION TO ARBITRARY LINEAR SPACES

The theory developed in this article can be applied not only to problems in which the opinions of the individuals are represented as subjective probability distributions. It can be applied, more generally, to problems in which the opinion F_i of each individual ($i = 1, \dots, k$) can be represented as a point in some fixed convex set in an arbitrary linear space, so that every convex linear combination of the form $\sum_{i=1}^n \pi_i F_i$ belongs to the set.

For example, F_i might represent a real-valued point estimate made by individual i of some parameter θ . If each individual then revises his estimate in accordance with (2.1), (3.1), and (3.3), this theory describes the process by which the group might reach agreement on a single point estimate of θ . More generally, F_i might be an m -dimensional vector representing the estimate made by individual i of an m -dimensional parametric vector θ .

8. CONCLUDING REMARKS

In the model that has been presented here, it is assumed that there is no possibility of learning whether the opinion of one individual is closer to the truth than that of another. In other words, it is assumed that no outside data, observations, or information about the value of θ is available. The only information available to an individual in the group at the beginning of the

process is the identity of the other members of the group. It is assumed that at the beginning, each individual i chooses the weights p_{ij} that he is going to use and he then continues to use these weights throughout the process. In practice, individual i might wish to change the weights that he assigns to the other individuals after he has learned their initial opinions, or after he has observed how much they change their opinions from stage to stage. This possibility has not, however, been studied here.

The theory presented here bears a resemblance to, but is quite distinct from, the Delphi technique for trying to reach agreement among a panel of experts (see, e.g., [2]). The Delphi technique is essentially an empirical procedure and is not based on any underlying mathematical model. However, it is typically applied iteratively in a sequence of stages. After each stage, the individuals are informed of the opinions of the others in the group and allowed to reassess their own opinions.

Because of the empirical nature of the Delphi technique, it differs from the theory presented here in that it provides no conditions under which the experts can be expected to reach agreement or for terminating the iterative process. Furthermore, in the most common version of the Delphi technique, although the individuals are informed of the totality of opinions of all the other individuals in the group after each stage, they are not told which person in the group holds each specific opinion. Indeed, they may not even be told the identities of the other members of the group.

Nevertheless, the theory presented in this article can be applied to this version of the Delphi technique, if each individual i assigns weight p_{ii} ($0 < p_{ii} < 1$) to his own opinion and equal weight $p_{ij} = (1 - p_{ii})/(k - 1)$ to the opinions of each of the other $k - 1$ individuals in the group. Furthermore, it follows immediately from Theorem 1 that this choice of weights leads to a consensus.

In this sense, the methods given here can be regarded as a formalization of the Delphi technique, a formalization in which a consensus is reached.

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