Social Influence. Reaching Consensus

Leonid E. Zhukov

School of Applied Mathematics and Information Science
National Research University Higher School of Economics

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НАЦИОНАЛЬНЫЙ ИССЛЕДОВАТЕЛЬСКИЙ
УНИВЕРСИТЕТ

Social Influence

Reaching a Consensus, Morris DeGroot 1974

Consensus = mutual agreement on a subject among group of people

- Group of people with opinions on the subject
- Can a common belief be reached?
- How long would it take?
- How each individual belief contribute to consensus?
- Which individuals have the most influence over final beliefs?

Model

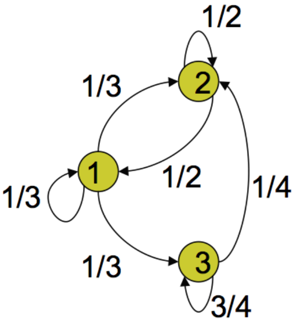
- Opinion $p_i(t) \in [0..1]$,
- T_{ij} is weight on the opinion of others, $i \to j$, how much i "listens to opnion"of j
- $\sum_j T_{ij} = 1$, raw-stochastic matrix
- Opinion update

$$p_i(t+1) = \sum_j T_{ij} p_j(t)$$

- Consensus is reached if

$$\exists \lim_{t \to \infty} p_i(t) = p^\infty$$

Example 1



$$T = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 \\ 0 & 1/4 & 3/4 \end{pmatrix}$$

Example 1

- Initial beliefs

$$p(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

- Updating

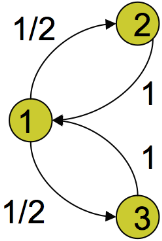
$$p(1) = Tp(0) = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 \\ 0 & 1/4 & 3/4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/2 \\ 0 \end{pmatrix}$$

$$p(2) = Tp(1) = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 \\ 0 & 1/4 & 3/4 \end{pmatrix} \begin{pmatrix} 1/3 \\ 1/2 \\ 0 \end{pmatrix} = \begin{pmatrix} 5/18 \\ 5/12 \\ 1/8 \end{pmatrix}$$

- Consensus

$$p(t) = Tp(t-1) = T^t p(0) \to \begin{pmatrix} 3/11 \\ 3/11 \\ 3/11 \end{pmatrix}$$

Example 2



$$T = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Example 2

- Initial beliefs

$$p(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

- Updating

$$p(1) = Tp(0) = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$p(2) = Tp(1) = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

- Consensus

$$\nexists \lim_{t \to \infty} T^t p(0)$$

Perron - Frobenius Theorem

Linear algebra, Graph theory, Markov chain theory.
Perron & Frobenius, 1912

Theorem

Let A be a square non-negative $A_{ij} \geq 0$ *irreducable matrix and* $\rho(A) = max_i(|\lambda_i|)$ *its spectral radius. Then*

1. $\rho(A) > 0$ *and* $\lambda_\rho = \rho(A)$ *is an eigenvalue with multiplicity one*

2. Left and right eigenvectors associated with λ_ρ *are strictly positive*

3. Any other eigenvalue $|\lambda| < \rho(A)$

4. $\min_i \sum_j A_{ij} \leq \rho(A) \leq \max_i \sum_j A_{ij}$

Reducible matrix:

$$P^T A P = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$$

Irreducable matrix = strongly connected graph

- Example 1:

$$T = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 \\ 0 & 1/4 & 3/4 \end{pmatrix}$$

Right: $\mathbf{Tp} = \lambda \mathbf{p}$, $\lambda = \{1, 0.5, 0.083\}$, $p_1 = \begin{pmatrix} 0.58 \\ 0.58 \\ 0.58 \end{pmatrix}$

Left: $\mathbf{pT} = \mathbf{p}\lambda$, $\lambda = \{1, 0.5, 0.083\}$, $p_1 = (0.46, 0.62, 0.62)$

- Example 2:

$$T = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Right: $\mathbf{Tp} = \lambda \mathbf{p}$, $\lambda = \{1, 0, -1\}$, $p_1 = \begin{pmatrix} 0.58 \\ 0.58 \\ 0.58 \end{pmatrix}$

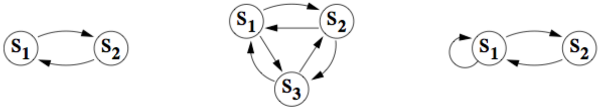
Left: $\mathbf{pT} = \mathbf{p}\lambda$, $\lambda = \{1, 0, -1\}$, $p_1 = (0.82, 0.41, 0.41)$

Stochastic matrices

- **T** - row-stochastic (right stocastic) matrix , i.e. $T_{ij} \geq 0$, $\sum_j T_{ij} = 1$
- Perron-Frobenius: if **A** - stochastic, then $\lambda_\rho = \rho(A) = 1$
- If **A** - row stochastic, it has right eigenvector $e = (1, 1, ...1)^T$,
  $Ae = \lambda e$, $\sum_j A_{ij} e_i = 1 \cdot e_i$, $\lambda = 1$
- **T**: - has largest eigenvalue $\lambda_1 = 1$
  $\mathbf{Tp} = \lambda \mathbf{p}$ - corresponding right (column) eigenvector
  $p_1 = e = (1, 1, ...1)^T = const$
  $\mathbf{pT} = \mathbf{p}\lambda$, - corresponding left (row) eigenvector all positive $p_{1i} > 0$

Aperiodic graph

- A graph is *aperiodic* if the greatest common divisor of the lengths of its cycles is one
  (there is no integer $k > 1$ that divides the length of every cycle of the graph)



- Convergence: stongly connected and aperiodic graph

Limiting belief

$p(t) = Tp(t-1)$, $t \to \infty$, $p(t) \to p^{\infty}$

$p^{\infty} = Tp^{\infty}$, $\lambda_1 = 1$, $p^{\infty} = \begin{pmatrix} c \\ \dots \\ c \end{pmatrix}$

$p^{\infty} = \lim_{t\to\infty} T^t p(0)$,

$$p^{\infty} = \begin{pmatrix} c \\ \dots \\ c \end{pmatrix} = \begin{pmatrix} T_{11} & \dots & T_{1n} \\ \dots & \dots & \dots \\ T_{n1} & \dots. & T_{nn} \end{pmatrix}^t \begin{pmatrix} p_1(0) \\ \dots \\ p_n(0) \end{pmatrix}$$

$$\lim_{t\to\infty} \begin{pmatrix} T_{11} & \dots & T_{1n} \\ \dots & \dots & \dots \\ T_{n1} & \dots. & T_{nn} \end{pmatrix}^t = \begin{pmatrix} s_1 & s_2 & \dots & s_n \\ s_1 & s_2 & \dots & s_n \\ s_1 & s_2 & \dots & s_n \end{pmatrix} = S = \begin{pmatrix} \vec{s} \\ \vec{s} \\ \vec{s} \end{pmatrix}$$

$$\begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 \\ 0 & 1/4 & 3/4 \end{pmatrix}^{20} = \begin{pmatrix} 0.27 & 0.36 & 0.36 \\ 0.27 & 0.36 & 0.36 \\ 0.27 & 0.36 & 0.36 \end{pmatrix}$$

Social influence

- $p^{\infty} = \lim_{t\to\infty} T^t p(0) = Sp(0)$
  $p^{\infty} = \lim_{t\to\infty} T^t p(1) = \lim_{t\to\infty} T^t Tp(0) = STp(0)$
  $(S - ST)p(0) = 0, \quad S = ST, \quad \begin{pmatrix} \vec{s} \\ \vec{s} \\ \vec{s} \end{pmatrix} = \begin{pmatrix} \vec{s} \\ \vec{s} \\ \vec{s} \end{pmatrix} T$
- s - left eigenevector T

$$\vec{s} = \vec{s} \cdot T$$

- Influence

$$p^{\infty} = \vec{s} \cdot p(0) = \sum_i s_i p_i(0)$$

$\vec{s} = (0.27, 0.36, 0.36)$

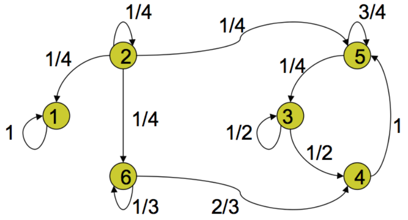
- Eigenvector centrality

$$A^T p = p$$

$$\vec{p}A = \vec{p}$$

Closed sets

- A set of nodes C is called a *closed set* if there is no direct link from the node in C to the node outside C (there is no $i \in C$ and $j \notin C$ such that $T_{ij} > 0$)



- T is convergent if and only if every set of nodes that is strongly connected and closed is aperiodic
- Every strongly connected closed and aperiodic set will reach own consensus

Model extensions

- Time-varying updates:

$$p(t) = \left[(1-\lambda(t))I + \lambda(t)\hat{T}\right] p(t-1)$$

- Time-varying weights on own beliefs:

$$p(t) = D\hat{T}p(t-1) + (I-D)p(0)$$

- Similar beliefs:

$$T_{ij}(p(t)) = \begin{cases} \frac{1}{n_i(p(t))}, & |p_i(t) - p_j(t)| < d, n_i = \#\{k : |p_i - p_k| < d\} \\ 0, & otherwise \end{cases}$$

- Close beliefs:

$$T_{ij}(p(t)) = \frac{e^{-\gamma_{ij}|p_i(t)-p_j(t)|}}{\sum_k e^{-\gamma_{ik}|p_i(t)-p_k(t)|}}$$

Social Influence Networks

Noah Friedkin 1991, ..,1999

- interpersonal influence for collective decisions
- modifying attitudes and opinions by interaction
- influence process in a group of N actors:

$$\mathbf{y}(t) = \mathbf{AWy}(t-1) + (\mathbf{I} - \mathbf{A})\mathbf{y}(1)$$

$\mathbf{y}(t)$ - vector of actors' opinion at time t
$\mathbf{y}(1)$ - vector of actors' initial opinion
$\mathbf{W}$ - N x N matrix of interpersonal influence
$\mathbf{A} = diag(a_{11}, .., a_{NN})$ - matrix of actors' suceptebilities to interpersonal influence

Assuming the process reaches an equilibrium, $lim_{t\to\infty}\mathbf{y}(t) = \mathbf{y}(\infty)$

$$\mathbf{y}(\infty) = \mathbf{AWy}(\infty) + (\mathbf{I} - \mathbf{A})\mathbf{y}(1)$$

Solution:

$$\mathbf{y}(\infty) = (\mathbf{I} - \mathbf{AW})^{-1}(\mathbf{I} - \mathbf{A})\mathbf{y}(1)$$

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