

Epidemic models

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НАЦИОНАЛЬНЫЙ ИССЛЕДОВАТЕЛЬСКИЙ
УНИВЕРСИТЕТ

Epidemic dynamics models

- Mathematical epidemiology
- W. O. Kermack and A. G. McKendrick, 1927
- Deterministic compartmental model (population classes) $\{S, I, T\}$
- $S(t)$ - susceptible, number of individuals not yet infected with the disease at time t
- $I(t)$ - infected, number of individuals who have been infected with the disease and are capable of spreading the disease.
- $R(t)$ - recovered, number of individuals who have been infected and then recovered from the disease, can't be infected again or to transmit the infection to others.
- Fully-mixing model
- Closed population (no birth, death, migration)
- Models: SI, SIS, SIR, SIRS,...

- $S(t)$ -susceptable , $I(t)$ - infected

$$S \longrightarrow I$$

$$S(t) + I(t) = N$$

- β - infection rate (on contact)
- Infection equation:

$$I(t + \delta t) = I(t) + \beta \frac{S(t)}{N} I(t) \delta t$$
$$\frac{dI(t)}{dt} = \beta \frac{S(t)}{N} I(t)$$

- Normalization: $i(t) = I(t)/N$, $s(t) = S(t)/N$
- Equations

$$\frac{ds(t)}{dt} = -\beta s(t)i(t)$$

$$\frac{di(t)}{dt} = \beta s(t)i(t)$$

$$s(t) + i(t) = 1$$

- Differential equation, $i(t = 0) = i_0$

$$\frac{di(t)}{dt} = \beta(1 - i(t))i(t)$$

- Solution:

$$\int \frac{di}{(1 - i)i} = \beta \int dt + C$$

$$\log \frac{i(t)}{1 - i(t)} = \beta t + C$$

- Logistic growth function:

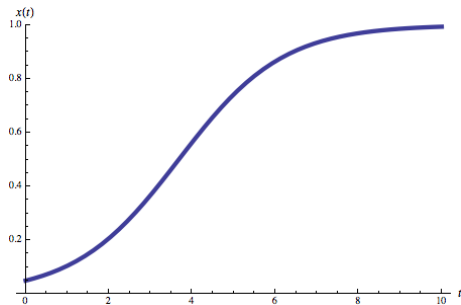
$$i(t) = \frac{i_0}{i_0 + (1 - i_0)e^{-\beta t}}$$

- Population:

$$I(t) = i(t)N$$

$$S(t) = (1 - i(t))N$$

Logistic growth function

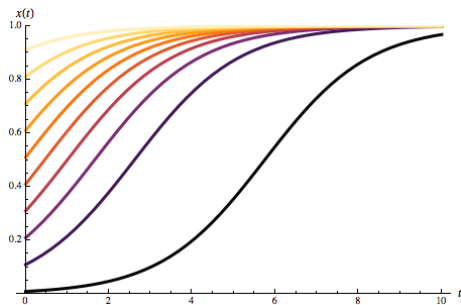


- $i_0 = 0.05, \quad \beta = 0.8$
- $t \rightarrow \infty$

$$I(\infty) = N$$

$$S(\infty) = 0$$

Logistic growth function



- $\beta = 0.8$

- $S(t)$ -susceptable , $I(t)$ - infected,

$$S \longrightarrow I \longrightarrow S$$

$$S(t) + I(t) = N$$

- β - infection rate (on contact), γ - recovery rate
- Infection equations:

$$\frac{ds}{dt} = -\beta si + \gamma i$$

$$\frac{di}{dt} = \beta si - \gamma i$$

$$s + i = 1$$

-

$$\frac{di}{dt} = (\beta - \gamma - i)i$$

- Solution

$$i(t) = \left(1 - \frac{\gamma}{\beta}\right) \frac{C}{C + e^{-(\beta-\gamma)t}}, \quad C = \frac{\beta i_0}{\beta - \gamma - \beta i_0}$$

- Limit $t \rightarrow \infty$

$$\beta > \gamma, \quad i(t) \rightarrow \left(1 - \frac{\gamma}{\beta}\right)$$

$$\beta < \gamma, \quad i(t) = i_0 e^{(\beta-\gamma)t} \rightarrow 0$$

- $\beta > \gamma$

$$I(\infty) = N \left(1 - \frac{\gamma}{\beta}\right)$$

$$S(\infty) = N \frac{\gamma}{\beta}$$

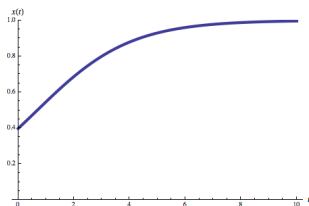
- $\beta < \gamma$

$$I(\infty) = 0$$

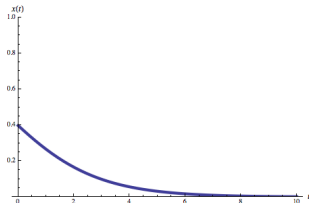
$$S(\infty) = N$$

Logistic function

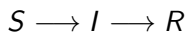
- $\beta > \gamma, \quad i(t) \rightarrow (1 - \frac{\gamma}{\beta})$



- $\beta < \gamma, \quad i(t) = i_0 e^{(\beta-\gamma)t} \rightarrow 0$



- $S(t)$ -susceptable , $I(t)$ - infected, $R(t)$ - recovered



$$S(t) + I(t) + R(t) = N$$

- β - infection rate, γ - recovery rate
- Infection equation:

$$\frac{ds}{dt} = -\beta si$$

$$\frac{di}{dt} = \beta si - \gamma i$$

$$\frac{dr}{dt} = \gamma i$$

$$s + i + r = 1$$

- Solution

$$\frac{ds}{dt} = -\beta s \frac{dr}{dt} \frac{1}{\gamma}$$

$$s = s_0 e^{-\frac{\beta}{\gamma} r}$$

$$\frac{dr}{dt} = \gamma i = \gamma(1 - s - r) = \gamma(1 - r - s_0 e^{-\frac{\beta}{\gamma} r})$$

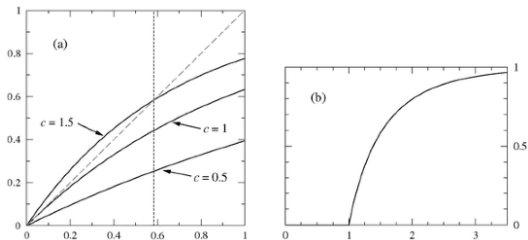
- Limits: $t \rightarrow \infty$, $r(\infty) \rightarrow \text{const}$, $\frac{dr}{dt} = 0$

$$1 - r = s_0 e^{-\frac{\beta}{\gamma} r}$$

- Initial conditions: $r(0) = 0$, $i(0) = c/N$, $s(0) = 1 - c/N \approx 1$

$$1 - r = e^{-\frac{\beta}{\gamma} r}$$

SIR model



$$r = 1 - e^{-cr}, \quad c = \frac{\beta}{\gamma}$$

$$(r)' = (1 - e^{-cr})', \quad \text{at } r = 0$$

critical value $c = 1$

- Critical point

$$\beta = \gamma$$

- Epidemic threshold

$$\beta > \gamma \quad , \quad r(\infty) = \text{const} > 0$$

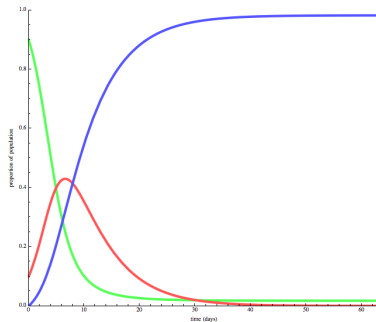
$$\beta < \gamma \quad , \quad r(\infty) \rightarrow 0$$

- Basic reproduction number

$$R_0 = \frac{\beta}{\gamma}$$

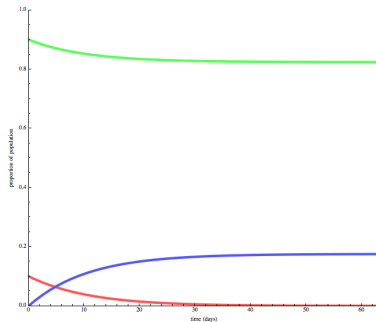
- $R_0 > 1$ - epidemics
 $R_0 < 1$ - no epidemics

SIR model



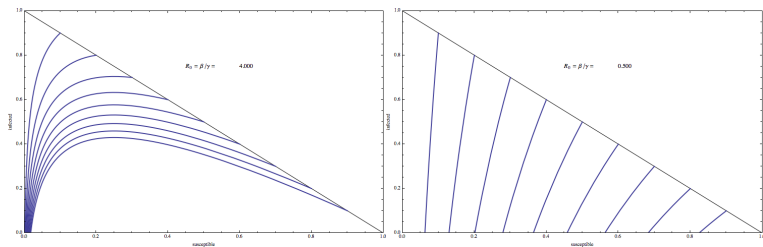
- $R_0 = 4$
- $i_0 = 0.1$

SIR model



- $R_0 = 0.5$
- $i_0 = 0.1$

Phase diagram



- A Contribution to the Mathematical Theory of Epidemics. , Kermack, W. O. and McKendrick, A. G. , Proc. Roy. Soc. Lond. A 115, 700-721, 1927.
- The Mathematics of Infectious Disease, Herbert W. Hethcote, SIAM Review, Vol. 42, No. 4, p. 599-653, 2000