

Network models: dynamical growth and small world

Leonid E. Zhukov

School of Applied Mathematics and Information Science
National Research University Higher School of Economics

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НАЦИОНАЛЬНЫЙ ИССЛЕДОВАТЕЛЬСКИЙ
УНИВЕРСИТЕТ

Empirical network features:

- Power-law (heavy-tailed) degree distribution
- Small average distance (graph diameter)
- Large clustering coefficient (transitivity)
- Giant connected component, hierarchical structure, etc

Generative models:

- Random graph model (Erdos & Renyi, 1959)
- Preferential attachment model (Barabasi & Albert, 1999)
- Small world model (Watts & Strogatz, 1998)

Growing (evolving with time) networks:

- Citation networks
- Collaboration networks
- Web
- Social networks

Stochastic growth model:

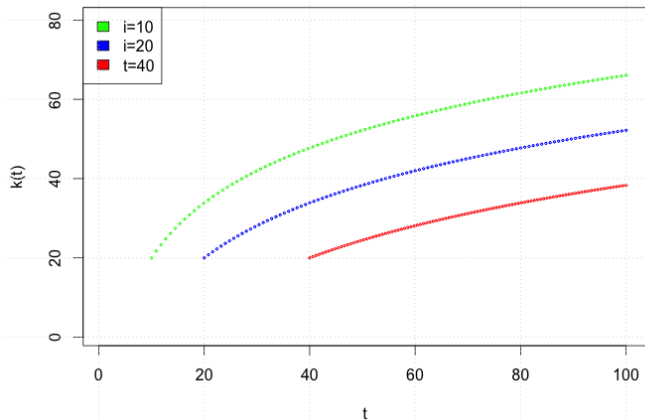
- $t = 0$, m unconnected nodes
- growth: on every time step $t = \{1, 2, 3, 4, \dots\}$ add a node with m edges
 $k_i(t = i) = m$
- attachment: form m edges with existing nodes uniformly at random,

Expected i -node degree at t $\langle k_i(t) \rangle$:

$$k_i(t) = m + \frac{m}{i+1} + \frac{m}{i+2} + \dots + \frac{m}{t}$$

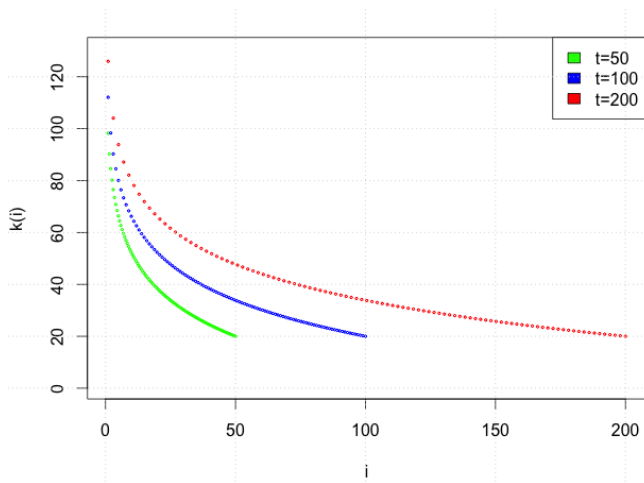
$$k_i(t) = m \left(1 + \log \left(\frac{t}{i} \right) \right)$$

Growing random graph



$$k_i(t) = m \left(1 + \log \left(\frac{t}{i} \right) \right), \quad m = 20, \quad i = 10, 20, 40, \quad t \geq i$$

Growing random graph



$$k_i(t) = m \left(1 + \log \left(\frac{t}{i} \right) \right), \quad m = 20, \quad t = 50, 100, 200, \quad i \leq t$$

Growing random graph

Find all nodes that at time t has degree less than k , $k_i(t) < k$?
(for example $k_i(t) < 40$)

$$\begin{aligned}k_i(t) &= m \left(1 + \log \left(\frac{t}{i} \right) \right) < k \\ \log \left(\frac{t}{i} \right) &< \frac{k}{m} - 1 \\ \frac{t}{i} &< e^{\frac{k-m}{m}} \\ i &> t e^{\frac{m-k}{m}}\end{aligned}$$

Fraction of nodes with degrees $k_i(t) < k$ (CDF):

$$F(k) = P(k_i(t) < k) = \frac{t-i}{t} = \frac{t - e^{\frac{m-k}{m}}}{t} = 1 - e^{\frac{m-k}{m}}$$

$$P(k) = \frac{d}{dk} F(k) = \frac{1}{m} e^{-\frac{k-m}{m}}, \quad k \geq m$$

Mean field approximation

Continues time approximation:

$$k_i(t + \delta t) = k_i(t) + \frac{m}{t} \delta t$$

Differential equation:

$$\frac{dk_i(t)}{dt} = \frac{m}{t}$$

with initial conditions: $k_i(t = i) = m$

Solution:

$$k_i(t) = m \left(1 + \log \left(\frac{t}{i} \right) \right)$$

Preferential attachment model

Barabasi and Albert, 1999

Dynamical growth model

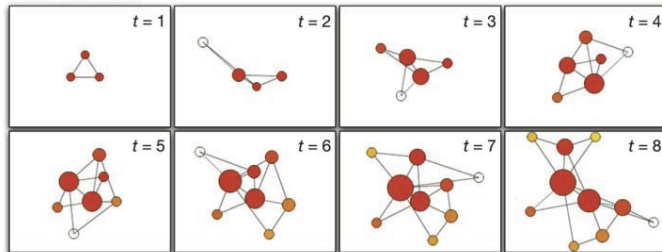
- $t = 0$, n_0 unconnected nodes
- growth: on every step add a node with m edges ($m \leq n_0$), $k_i(i) = m$
- Preferential attachment: probability of linking to existing node is proportional to the node degree k_i

$$\Pi(k_i) = \frac{k_i}{\sum_i k_i}$$

after t steps: $n_0 + t$ nodes, mt edges

Preferential attachment model

Scale-Free Model



Barabasi, 1999

Preferential attachment

Continues time, mean field approximation:

$$k_i(t + \delta t) = k_i(t) + m\Pi(k_i)\delta t$$

$$\frac{dk_i(t)}{dt} = m\Pi(k_i) = m\frac{k_i}{\sum_i k_i} = \frac{mk_i}{2mt}$$

$$\frac{dk_i(t)}{dt} = \frac{k_i(t)}{2t}$$

initial conditions: $k_i(i) = m$

Solution:

$$k_i(t) = m \left(\frac{t}{i} \right)^{1/2}$$

Preferential attachment

Time evolution of a node degree

$$k_i(t) = m \left(\frac{t}{i} \right)^{1/2}$$

Nodes with $k_i(t) < k$:

$$m \left(\frac{t}{i} \right)^{1/2} < k$$
$$i > \frac{m^2}{k^2} t$$

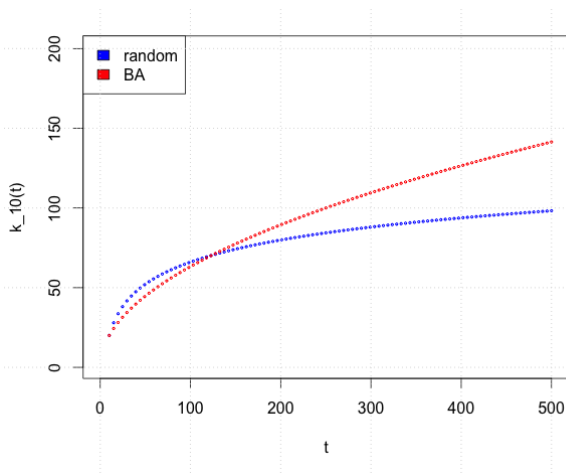
Fraction of nodes with $k_i(t) < k$ (CDF):

$$F(k) = P(k_i(t) < k) = \frac{n_0 + t - i}{n_0 + t} = \frac{n_0 + t - m^2 t / k^2}{n_0 + t} \approx 1 - \frac{m^2}{k^2}$$

Distribution function:

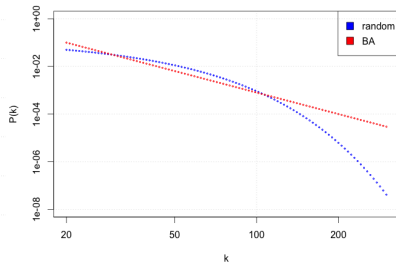
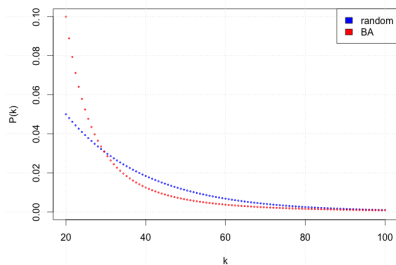
$$P(k) = \frac{d}{dk} F(k) = \frac{2m^2}{k^3}$$

Dynamic growth



$$BA : k_i(t) = m \left(\frac{t}{i} \right)^{1/2}, \quad RG : k_i(t) = m \left(1 + \log \left(\frac{t}{i} \right) \right)$$

Dynamic growth



$$BA : P(k) = \frac{2m^2}{k^3}, \quad RG : P(k) = \frac{1}{m} e^{-\frac{k-m}{m}}$$

- Power law distribution function:

$$P(k) = \frac{2m^2}{k^3}$$

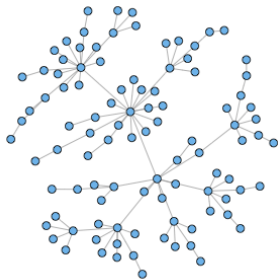
- Average path length (analytical result) :

$$\langle L \rangle \sim \log(N) / \log(\log(N))$$

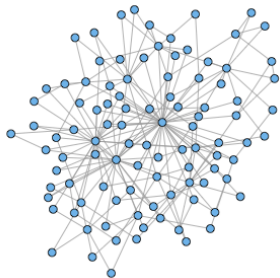
- Clustering coefficient (numerical result):

$$C \sim N^{-0.75}$$

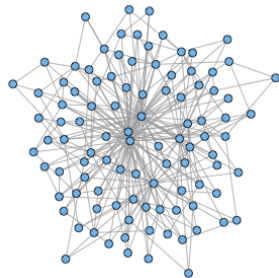
Preferential attachment model



$m = 1$



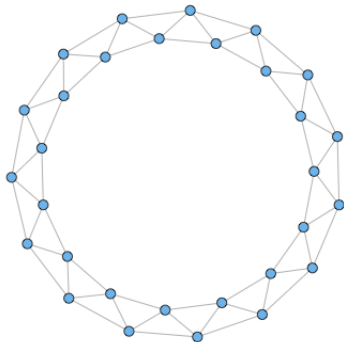
$m = 2$



$m = 3$

Small world

Motivation: keep high clustering, get small diameter



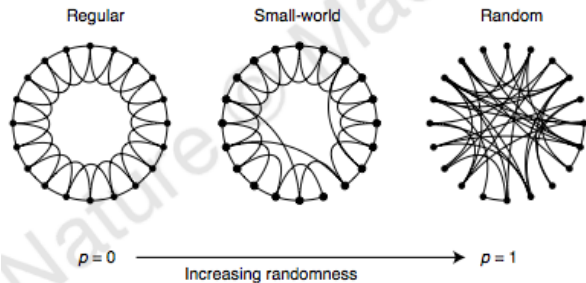
Clustering coefficient $C = 1/2$

Graph diameter $d = 8$

Watts and Strogatz, 1998

Single parameter model, interpolation between regular lattice and random graph

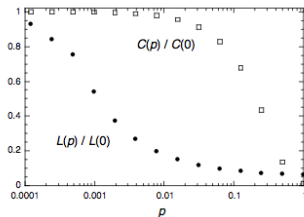
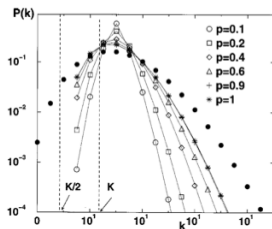
- start with regular lattice with N nodes, K nearest neighbours (node degree)
- randomly connect with other nodes with probability p , forms $pNK/2$ "long distance" connections
- $p = 0$ regular lattice, $p = 1$ random graph



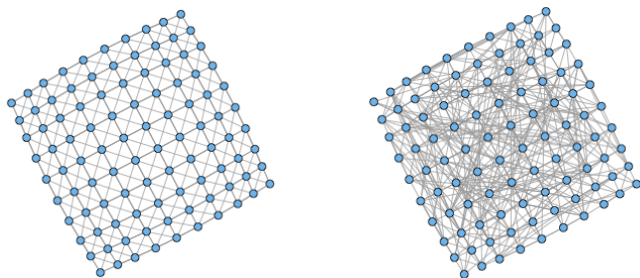
Watts, 1998

Small world model

- Node degree distribution:
Poisson like
- Ave. path length $\langle L(p) \rangle$:
 $p \rightarrow 0$, ring lattice, $\langle L(0) \rangle = 2n/k$
 $p \rightarrow 1$, random graph, $\langle L(1) \rangle = \log(n)/\log(k)$
- Clustering coefficient $C(p)$:
 $p \rightarrow 0$, ring lattice, $C(0) = 3/4 = \text{const}$
 $p \rightarrow 1$, random graph, $C(1) = k/n$



Small world model



20% rewiring:

ave. path length = 3.58 \rightarrow ave. path length = 2.32

clust. coeff = 0.49 \rightarrow clust. coeff = 0.19

Model comparison

	Random	BA model	WS model	Empirical networks
$P(k)$	$\frac{\lambda^k e^{-\lambda}}{k!}$	k^{-3}	poisson like	power law
C	$\langle k \rangle / N$	$N^{-0.75}$	const	large
$\langle L \rangle$	$\frac{\log(N)}{\log(\langle k \rangle)}$	$\frac{\log(N)}{\log \log(N)}$	$\log(N)$	small

- Collective dynamics of 'small-world' networks. Duncan J. Watts and Steven H. Strogatz. *Nature* 393 (6684): 440–442, 1998
- Emergence of Scaling in Random Networks, AL Barabasi and R. Albert, *Science* 286, 509–512, 1999