Lecture 6

$$P_{ij}(t) = P\{X(s+t)=j \mid X(s)=i\}$$

$$P_{ij}(0) = \delta_{ij}, \quad \sum_j P_{ij}(t) = 1$$

$$q_{ij} = \lim \frac{P_{ij}(t) - P_{ij}(0)}{t} = \frac{dP_{ij}}{dt}\Big|_{t=0} \qquad i \neq j$$

$$q_i = \lim \frac{1 - P_{ii}(t)}{t} = -\frac{dP_{ii}}{dt}\Big|_{t=0}$$

$$q_{ii} = -q_i$$

$$Q = \begin{pmatrix} q_{00} & \boxed{\;\; q_{0S}} \\ q_{S0} & q_{SS} \end{pmatrix}$$

$$\sum_j q_{ij} = \sum_{j \neq i} q_{ij} + q_{ii} = q_i + q_{ii} = 0$$

$$q_i = \sum_{j \neq i} q_{ij}$$

$$P^{n+m} = P^n \cdot P^m$$

$$P(t+s) = P(t)\, P(s)$$

$$P_{ij}(t+s) = \sum_{k=0}^{\infty} P_{ik}(t) P_{kj}(s)$$

$$P_{ij}(h+t) - P_{ij}(t) = \sum_k P_{ik}(h) P_{kj}(t) - P_{ij}(t)$$

$$= \sum_{k \neq i} P_{ik}(h) P_{kj}(t) + P_{ii}(h) P_{ij}(t)$$

$$P_{ij}(t) = \sum_{k \neq i} P_{ik}(h) P_{kj}(t) - (1 - P_{ii}(h)) P_{ij}(t)$$

$$\lim_{h \to 0} \frac{P_{ij}(t+h) - P_{ij}(t)}{h} = \lim_{h \to 0} \sum_{i \neq k} \frac{P_{ik}(h)}{h} P_{kj}(t) - \frac{1 - P_{ii}(h)}{h} P_{ij}(t)$$

q_{ik} $\qquad$ q_i

$$\boxed{\frac{dP_{ij}(t)}{dt} = \sum_{k \neq i} q_{ik} P_{kj}(t) - q_i P_{ij}(t)}$$

$0 \xrightarrow{} 1$

$\frac{1}{\lambda}$ $\quad q_{01} = \lambda$, $\quad q_{10} = 0$

$$Q = \begin{pmatrix} -\lambda & \lambda \\ 0 & 0 \end{pmatrix}$$

$$\frac{d}{dt} P_{00}(t) = q_0 (P_{10}(t) - q_0 P_{00}(t) = \lambda P_{10}(t) - \lambda P_{00}(t)$$

$$\frac{dP_{10}(t)}{dt} = q_{10} P_{00}(t) - q_1 P_{10}(t) = 0\, P_{00}(t) - 0\, P_{10}(t) = 0$$

$$\left.\begin{array}{l} P_{10}(t) = \text{const} \\ P_{10}(0) = \delta_{10} = 0 \end{array}\right| \Rightarrow P_{10}(t) = 0$$

$$\frac{dP_{00}(t)}{dt} = -\lambda P_{00}(t)$$

$$P_{00}(t) = c e^{-\lambda t}$$

$$P_{00}(0) = \delta_{00} = 1$$

$$P_{00}(t) + P_{01}(t) = 1$$

$$P_{01}(t) = 1 - P_{00}(t) = 1 - e^{-\lambda t}$$

$$\boxed{P(t) = \begin{pmatrix} e^{-\lambda t} & 1 - e^{-\lambda t} \\ 0 & 1 \end{pmatrix}}$$

$0 \xrightarrow{\lambda} 1$

$$q_{ij}$$

$$q_{ii} = -q_i$$

$$q_{ij} = \lim \frac{P_{ij}(t)}{t}$$

$$q_{ii} = \lim \frac{P_{ij}(t) - 1}{t}$$

$$Q = \lim_{t \to 0} \frac{P(t) - I}{t}$$

$$\boxed{\frac{dP(t)}{dt} = P(t)\, Q}$$

$$\boxed{\frac{dP(t)}{dt} = Q\, P(t)}$$

$$P(0) = I$$

$$P(t) = e^{Qt} =$$

$$= I + Qt + \frac{(Qt)^2}{2!} + \frac{(Qt)^3}{3!} + \dots =$$

$$= \sum_{n=0}^{\infty} \frac{(Qt)^n}{n!} = e^{Qt}$$

$$\frac{dP}{dt} = e^{Qt} Q$$

$$\sum_{n=0}^{\infty} n \frac{(Qt)^{n-1}}{n!} Q \rightarrow \sum_{n=0}^{\infty} \frac{(Qt)^n}{n!} Q$$

$$P(t) = e^{Qt}$$

$$Q = R \cdot D \cdot R^{-1}$$

$$\boxed{QR = RD}$$

$$P(t) = \boxed{e^{Qt}} = e^{RDR^{-1}t} =$$

$$= \sum \frac{t^n (RDR^{-1})^n}{n!} = \sum_n \frac{t^n (RDR^{-1}RDR^{-1}RDR^{-1})}{n!}$$

I

$$= R \sum_n \frac{t^n D^n}{n!} R^{-1} = \boxed{R e^{Dt} R^{-1}}$$

$$P(t) = e^{Qt} = R \begin{pmatrix} e^{t\lambda_1} & & \\ & \ddots & \\ & & e^{t\lambda_S} \end{pmatrix} R^{-1}$$

$0 \xrightarrow{\lambda} 1$

$$Q = \begin{pmatrix} -\lambda & \lambda \\ 0 & 0 \end{pmatrix}$$

$$\boxed{P(t) = R \begin{pmatrix} e^{\varkappa_1 t} & 0 \\ 0 & e^{\varkappa_2 t} \end{pmatrix} R^{-1}}$$

$$\boxed{QR = RD}$$

$$QR = R \begin{pmatrix} \varkappa_1 & 0 \\ 0 & \varkappa_2 \end{pmatrix}$$