Link Analysis

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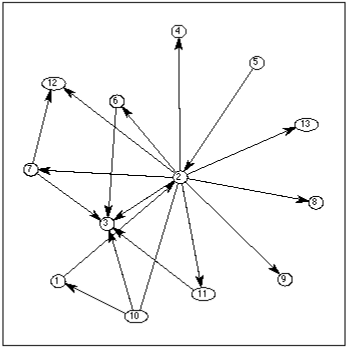
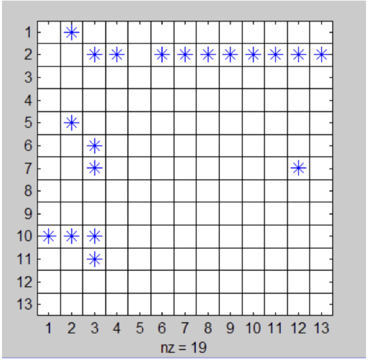
Graph $G(n, m)$ and adjacency matrix A_{ij}, edge $i \to j$

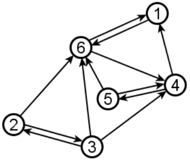




Leo Katz, 1953.

Take into account status (prestige) of directly connected actors

$$p_i \leftarrow \sum_{j \in N(i)} p_j = \sum_j A_{ji} p_j$$



$$p_i = \sum_j A_{ji} p_j$$

$$\mathbf{p} = \mathbf{A}^T \mathbf{p}$$

$$(\mathbf{I} - \mathbf{A}^T)\mathbf{p} = 0$$

Nontrivial solution only if $det(\mathbf{I} - \mathbf{A}^T) = 0$. Need to constraint matrix

Status or Rank Prestige

Leo Katz, 1953

An actor gives equal parts of its prestige to all nearest neighbours

$$p_i \leftarrow \sum_j A_{ji} \frac{p_j}{k_{out}(j)} = \sum_j \frac{A_{ji}}{k_{out}(j)} p_j$$

$$\mathbf{p} = (\mathbf{D}^{-1}\mathbf{A})^T \mathbf{p}$$

where $\mathbf{D}_{ii} = max(k_i, 1)$

$\mathbf{D}^{-1}\mathbf{A}$ - stochastic matrix, $\sum_j (\mathbf{D}^{-1}\mathbf{A})_{ij} = 1$, *guaranteed* $\lambda_{max} = 1$

$$(\mathbf{D}^{-1}\mathbf{A})^T \mathbf{p} = \lambda \mathbf{p}$$

Eigenvector Centrality

Phillip Bonacich, 1972.

$$c_i \leftarrow \sum_j A_{ji} c_j$$

$$c_i = \frac{1}{\kappa} \sum_j A_{ji} c_j$$

$$\kappa \mathbf{c} = \mathbf{A}^T \mathbf{c}$$

$$(\kappa \mathbf{I} - \mathbf{A}^T)\mathbf{p} = 0$$

Nontrivial solution only when $det(\kappa \mathbf{I} - \mathbf{A}^T) = 0$. Eigenvalue problem.
Choose eigenvector corresponding do maximum eigenvalue:
$\kappa_{max} = \kappa_1$, $\mathbf{c} = \mathbf{c}_1$

Status or Rank Prestige. Eigenvector Centrality

Phillip Bonacich, 1987.
Parametrized centrality measure $c(\alpha, \beta)$

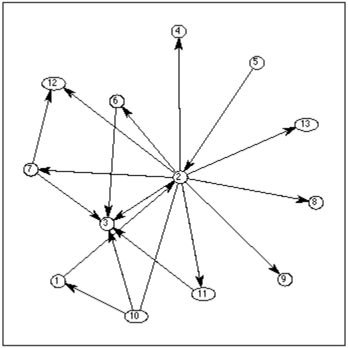
$$c_i = \sum_j (\alpha + \beta c_j) A_{ji}$$

$$\mathbf{c} = \alpha \mathbf{A}^T \mathbf{e} + \beta \mathbf{A}^T \mathbf{c}$$

$$\mathbf{c} = \alpha (\mathbf{I} - \beta \mathbf{A}^T)^{-1} \mathbf{A}^T \mathbf{e}$$

α - found from normalizaion $||\mathbf{c}||_2 = \sum c_i^2 = 1$
β - parameter, degree and direction of dependence on others

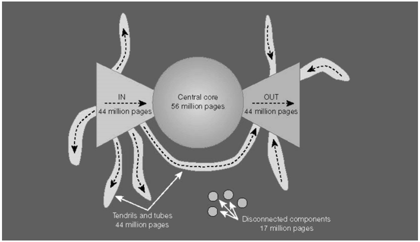
Graph $G(n, m)$

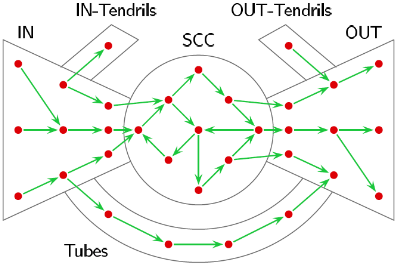


- zero out degree nodes, $k_{out}(i) = 0$
- zero in degree nodes, $k_{in}(i) = 0$

Real World

Bow tie structure of the web





Perron-Frobenius Theorem

Oscar Perron, 1907, Georg Frobenius,1912.
Eigenvalue problem:

$$\mathbf{Pp} = \lambda \mathbf{p}$$

Perron-Frobenius theorem: Real square matrix with positive entries

- stochastic (non-negative and rows sum up to one)
- irreducible (strongly connected graph)
- aperiodic

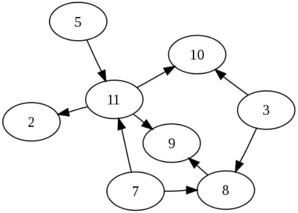
then unique largest eigenvalue $\lambda_{max} = 1$, with positive left eigenvector and power iterations coneverges to it. Solution satisfies $|\mathbf{p}|_1 = 1$
Stationary distribution of Markov chain

PageRank

Sergey Brin and Larry Page, 1998

Transition matrix:

$$\mathbf{P} = \mathbf{D}^{-1}\mathbf{A}$$



Stochastic matrix:

$$\mathbf{P}' = \mathbf{P} + \frac{\mathbf{d}\mathbf{e}^T}{n}$$

PageRank matrix:

$$\mathbf{P}'' = \alpha\mathbf{P}' + (1-\alpha)\frac{\mathbf{e}\mathbf{e}^T}{n}$$

Eigenvalue problem (choose solution with $\lambda = 1$):

$$\mathbf{P}''^T\mathbf{p} = \lambda\mathbf{p}$$

Notations:

e - unit column vector, **d** - absorbing nodes indicator vector (column)

PageRank computations

- Eigenvalue problem

$$\left[\alpha\left((\mathbf{D}^{-1}\mathbf{A})^T + \frac{\mathbf{e}\mathbf{d}^T}{n}\right) + (1-\alpha)\frac{\mathbf{e}\mathbf{e}^T}{n}\right]\mathbf{p} = \lambda\mathbf{p}$$

- Power iterations

$$\mathbf{p} \leftarrow \alpha(\mathbf{D}^{-1}\mathbf{A})^T\mathbf{p} + \alpha\frac{\mathbf{e}}{n}(\mathbf{d}^T\mathbf{p}) + (1-\alpha)\frac{\mathbf{e}}{n}(\mathbf{e}^T\mathbf{p})$$

$$\mathbf{p} \leftarrow \frac{\mathbf{p}}{||\mathbf{p}||}$$

- Sparse linear system ($\lambda = 1$, $||p||_1 = 1$)

$$\left[\mathbf{I} - \alpha\left((\mathbf{D}^{-1}\mathbf{A})^T + \frac{\mathbf{e}\mathbf{d}^T}{n}\right)\right]\mathbf{p} = (1-\alpha)\frac{\mathbf{e}}{n}$$

Hubs and Authorities

HITS, Jon Kleinberg, 1999

Citation networks. Reviews vs original research (authoritative) papers

- authorities, contain useful information, a_i
- hubs, contains links to authorities, h_i

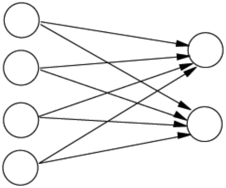
Mutual recursion

- Good authorities reffered by good hubs

$$a_i \leftarrow \sum_j A_{ji} h_j$$

- Good hubs point to good authorities

$$h_i \leftarrow \sum_j A_{ij} a_j$$



HITS

System of linear equations

$$\begin{aligned} \mathbf{a} &= \alpha \mathbf{A}^T \mathbf{h} \\ \mathbf{h} &= \beta \mathbf{A} \mathbf{a} \end{aligned}$$

Symmetric eigenvalue problem

$$\begin{aligned} (\mathbf{A}^T \mathbf{A})\mathbf{a} &= \lambda \mathbf{a} \\ (\mathbf{A}\mathbf{A}^T)\mathbf{h} &= \lambda \mathbf{h} \end{aligned}$$

where eigenvalue $\lambda = (\alpha\beta)^{-1}$

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